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Some Common Fixed Point Theorems in Fuzzy 2-Metric Spaces

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Abstract. The aim of this study was to prove some common fixed point theorems in fuzzy 2-metric spaces by removing the assumption of continuity, relaxing the compatibility or compatibility of type (α) or compatibility of type (β) to weak compatibility, and replacing the completeness of the space with a set of alternative conditions.

Keywords: fuzzy metric spaces, coincidence point, common fixed point, compatible maps, weakly compatible maps

Introduction

The concept of fuzzy sets was initially introduced by Zadeh (1965). Since then, for the purpose of using this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. Kaleva and Seikkala (1984), Deng (1982), Kramosil and Michalek (1975), and Erceg (1970) have particularly introduced the concept of fuzzy metric spaces in different ways. Recently, many authors have also studied the fixed point theory in these fuzzy metric spaces (Sharma and Deshpande, 2003; 2002; Chugh and Kumar, 2001; Jung *et al.,* 1996; 1994; Fang, 1994; Hadzic, 1989; Grabiec, 1988; Badard, 1984; Singh and Ram, 1982). Mishra *et al.* (1994) proved common fixed point theorems on complete fuzzy metric spaces, which generalized, extended and fuzzified several known fixed point theorems for contractive type maps on metric and other spaces. They assumed continuity of one map in each of the two pairs of compatible maps and also the commutativity of continuous maps. Cho (1997) and Jung *et al.* (1994) extended and generalized several fixed point theorems on metric spaces, Menger probabilistic metric spaces and uniform spaces, and proved common fixed point theorems on complete fuzzy metric spaces. The results of Cho (1997) were extended by Sharma and Deshpande (2002), and Sharma (2001). Cho (1997), Hadzic (1989), Jungck *et al.* (1983), and Singh and Kasahara (1983) have proved the common fixed point theorems for mappings under the condition of continuity and compatibility of type (α) in complete fuzzy metric, probabilistic metric, and metric spaces. In this paper, the assumption of continuity was removed, relaxing compatibility or compatibility of type (α) or compatibility of type (β) to weak compatibility, and the completeness of space was replaced with a set of alternative conditions. The assumption of commutativity of continuous maps was also removed in the case of two pairs of maps. The results of Sharma and Deshpande (2003) have been extended.

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Definitions

Given below are definitions used in the paper.

Definition 1.1. A binary operation $*$: $[0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm, if ([0,1], $*$) is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \le a_2 * b_2 * c_2$, whenever $a_1 \le a_2$ and $b_1 \le b_2$ and $c_1 \le c_2$ for all $a_1\ a_2$, $b_1\ b_2$, $c_1\ c_2$ are in [0, 1] (Sharma, 2002).

Definition 1.2. The 3-tuple (X, M,$*$) is called a fuzzy 2-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^3 \times (0, \infty)$, satisfying the following conditions: for all $x, y, z, u \in X$, and $t_1, t_2, t_3 > 0$ (Sharma, 2002).

(FM-1) $M(x, y, z, 0) = 0$,

(FM-2) $M(x, y, z, t) = 1$, $t > 0$ and when at least two of the three points are equal,

(FM-3) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) = ...$ (symmetry about three variables),

(FM-4) $M(x, y, z, t_1 + t_2 + t_3) \ge M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$ (this corresponds to tetrahedron inequality in 2-metric space),

(FM-5) $M(x, y, z,.) : [0, \infty) \to [0, 1]$ is left continuous.

Example 1.1. Let (X, d) metric space define $a * b = ab$ or $a * b = \min\{a, b\}$ and for all $x, y \in X$ and $t > 0$

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

then:

(X, M,$*$) is a fuzzy metric space

This is called fuzzy metric M induced by metric d, the standard fuzzy metric.

Definition 1.3. Let (X, M,*) be a fuzzy 2-metric space:

then:

- a sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$, if
$\lim_{n\to\infty} M(x_n, x, a, t) = 1$

for all $a \in X$ and $t > 0$

- a sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if
$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1$

for all $a \in X$ and $t > 0$, $p > 0$

- a fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete

Remark 1.1. Since * is continuous, it follows from ***(FM-4)*** that the limit of the sequence in fuzzy 2-metric space is uniquely determined (Sharma, 2002).

Let (X, M, *) be a fuzzy 2-metric space with the following condition:

(FM – 6) $\lim_{n\to\infty} M(x, y, z, t) = 1$

for all $x, y, z \in X$

Definition 1.4. A pair of mappings A and S is called a weakly compatible pair if they commute at coincidence points (Jungck and Rhoades, 1998).

Example 1.2. Define $A, S : [0, 3] \to [0, 3]$ by:

$$A(x) = \begin{cases} x, \text{ if } x \in [0, 1] \\ 3, \text{ if } x \in [1, 3] \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 3 - x, \text{ if } x \in [0, 1] \\ 3, \text{ if } x \in [1, 3] \end{cases}$$

then:

for any $x \in [1, 3]$, $ASx = SAx$, showing that A, S are weakly compatible maps on [0, 3].

Example 1.3. Let $X = [0, 2]$ with the metric d defined by:

$d(x, y) = |x - y|$, then for each $t \in (0, \infty)$ define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad M(x, y, 0) = 0, x, y \in X$$

Clearly, M(X, M,*) is a fuzzy metric space on X, where * is defined by $a*b = ab$ or $a*b = \min\{a, b\}$.

Define $A, B: X \to X$ by $Ax = x$, if $x \in [0, 1/3)$, $Ax = 1/3$, if $x \geq 1/3$ and $Bx = x/(1 + x)$, $x \in [0, 2]$.

Consider the sequence $\{ x_n = ½ + 1/n; n \geq 1 \}$ in X

then:

$\lim_{n\to\infty} Ax_n = 1/3$ and $\lim_{n\to\infty} Bx_n = 1/3$

but

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \frac{t}{t + |1/3 - 1/4|} \neq 1$$

thus:

A and B are non-compatible, but A and B are commuting at their coincidence point x = 0, that is, weakly compatible at x = 0

also

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = \frac{t}{t + |1/3 - 1/4|} \neq 1$$

and

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = \frac{t}{t + |1/4 - 1/3|} \neq 1$$

further

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = \frac{t}{t + |1/3 - 1/4|} \neq 1$$

thus:

A and B are not compatible of type (β)

In view of this example, it is observed that:

- weakly compatible maps need not be compatible
- weakly compatible maps need not be compatible of type (α)
- weakly compatible maps need not be compatible of type (β)

Lemma 1.1. For all $x, y, z \in X$, M(x, y, z, .) is non-decreasing (Sharma, 2002).

Lemma 1.2. Let $\{y_n\}$ be a sequence in a fuzzy 2-metric space (X, M, *) with the condition ***(FM-6)***, if there exists a number $k \in (0, 1)$, such that:

$M(y_{n+2}, y_{n+1}, a, kt) \geq M(y_{n+1}, y_n, a, t)$

for all $t > 0$ and $a \in X$ and n = 1, 2,…, then $\{y_n\}$ is a Cauchy sequence in X (Sharma, 2002).

Lemma 1.3. If for all $x, y, a \in X$, $t > 0$ and for a number $k \in (0, 1)$

$M(x, y, a, kt) \geq M(x, y, a, t)$

then, x = y (Sharma, 2002).

Results and Discussion

Sharma and Deshpande (2003) proved the following:

Theorem A. Let (X, M, *) be a fuzzy metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM- 6)***. Let A, B, S, T be mappings from X into itself, such that:

(1) $A(X) \subset T(X),\ B(X) \subset S(X)$

(2) there exists a constant $k \in (0, 1)$, such that:

$M(Ax, By, kt) \geq M(Ty, By, t) * M(Sx, Ax, t) * M(Sx, By, \alpha t)$
$* M[Ty, Ax, (2-\alpha)t] * M(Ty, Sx, t)$

for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$

(3) one of A(X), B(X), S(X), or T(X) is a complete subspace of X,

then:

(i) A and S have a coincidence point
(ii) B and T have a coincidence point

further, if

(4) the pairs {A, S} and {B, T} are weakly compatible

then:

(iii) A, B, S and T have a unique fixed point in X

Theorem A for fuzzy 2-metric space, has been proved. The following are proved as:

Theorem 2.1. Let (X, M, *) be a fuzzy 2-metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM-6)***. Let A, B, S and T be mappings from X into itself, such that:

(2.1) $A(X) \subset T(X)$, $B(X) \subset S(X)$

(2.2) there exists a constant $k \in (0, 1)$

such that:

$M(Ax, By, a, kt) \geq M(Ty, By, a, t) * M(Sx, Ax, a, t) * M(Sx, By, a, \alpha t)$
$* M[Ty, Ax, a, (2-\alpha)t] * M(Ty, Sx, a, t)$

for all $x, y, a \in X$, $\alpha \in (0, 2)$ and $t > 0$

(2.3) one of A(X), B(X), S(X) or T(X) is a complete subspace of X

then:

(i) A and S have a coincident point
(ii) B and T have a coincident point

further, if

(2.4) the pairs {A, S} and {B, T} are weakly compatible

then:

(iii) A, B, S and T have a unique fixed point in X

Proof. By ***(2.1)***, since $A(X) \subset T(X)$, so for any arbitrary $x_0 \in X$, there exists a point $x_1 \in X$, such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, a point $x_2 \in X$ can be chosen, such that $Bx_1 = Sx_2$, and so on. Inductively, a sequence $\{y_n\}$ in X can be defined as:

$y_{2n} = Tx_{2n+1} = Ax_{2n}$, and
$y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$ for $n = 0, 1, 2, \ldots$

By ***(2.2)*** for all $t > 0$ and $\alpha = 1 - q$, with $q \in (0,1)$, it would be:

$M(Ax_{2n+2}, Bx_{2n+1}, a, kt) \geq M(Tx_{2n+1}, Bx_{2n+1}, a, t) * M(Sx_{2n+2}, Ax_{2n+2}, a, t) * M(Sx_{2n+2}, Bx_{2n+1}, a, \alpha t) * M[Tx_{2n+1}, Ax_{2n+2}, a, (2-\alpha) t] * M(Tx_{2n+1}, Sx_{2n+2}, a, t)$

$M(y_{2n+2}, y_{2n+1}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n+1}, y_{2n+1}, a, \alpha t) * M[y_{2n}, y_{2n+2}, a, (2-\alpha) t] * M(y_{2n}, y_{2n+1}, a, t) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * 1 * M[y_{2n}, y_{2n+2}, a, (1+q) t] * M(y_{2n}, y_{2n+1}, a, t)$

On the lines of Sharma (2002) it is:

$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n}, y_{2n+2}\ a, tq + t/2 + t/2)$
$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n}, y_{2n+2}, y_{2n+1}, qt) * M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$
$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_0, y_2, y_1, t/3q^{2n}) * M(y_0, y_1, y_1, t/3q^{2n}) * M(y_1, y_2, y_1, t/3q^{2n}) * M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$

thus:

since $M(y_0, y_2, y_1, t/3q^{2n}) \to 1$ as $n \to \infty$

it is:

(2.5) $M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t)$

similarly, it is:

(2.6) $M(y_{2n+2}, y_{2n+3}, a, kt) \geq M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n+2}, y_{2n+3}, a, t)$

from ***(2.5)*** and ***(2.6)***, it follows that:

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t)$

for $n = 1, 2, \ldots$, and also for positive integer n, p

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t/k^P)$

thus:

since $M(y_{n+1}, y_{n+2}, a, t/k^P) \to 1$ as $p \to \infty$

it is:

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t)$

so, by ***Lemma 1.2,*** $\{y_n\}$ is a Cauchy sequence in X.

Now, suppose S(X) is complete, note that S(X) contained the sub-sequence $\{y_{2n+1}\}$ in S(X) and has a limit in S(X), it is called z.

Let $u = S^{-1}z$, thus, $Su = z$. The fact will be used that the sub-sequence $\{y_{2n}\}$ also converges to z

by ***(2.2)*** with $\alpha = 1$, it is:

$M(Au, y_{2n+1}, a, kt) = M(Au, Bx_{2n+1}, a, kt)$
$\geq M(Tx_{2n+1}, Bx_{2n+1}, a, t) * M(Su, Au, a, t) * M(Su, Bx_{2n+1}, a, t) * M(Tx_{2n+1}, Au, a, t) * M(Tx_{2n+1}, Su, a, t)$
$= M(y_{2n}, y_{2n+1}, a, t) * M(Su, Au, a, t) * M(Su, y_{2n+1}, a, t) * M(y_{2n}, Au, a, t) * M(y_{2n}, Su, a, t)$

which implies that as $n \to \infty$

$M(Au, z, a, kt) \geq 1 * M(z, Au, a, t) * 1 * M(z, Au, a, t) * 1$
$\geq M(Au, z, a, t)$

Therefore, by ***Lemma 1.3,*** $Au = z$

thus:

$Au = z = Su$, i.e., u is a coincidence point of A and S. This proves ***(i)***.

Since $A(X) \subset T(X)$, $Au = z$ implies that $z \in T(X)$. Let $v = T^{-1}z$, then $Tv = z$. It can be easily verified by using similar arguments of the previous part of the proof that $Bv = z$

thus:

$Bv = z = Tv$, i.e., v is a coincidence point of B and T. This proves ***(ii)***.

If it is assumed that T(X) is complete, then argument analogous to the previous completeness argument establishes ***(i)*** and ***(ii)***. The remaining two cases pertain essentially to the previous cases. Indeed, if B(X) is complete, then by ***(2.1)***, $z \in B(X) \subset S(X)$. Similarly, if A(X) is complete, then $z \in A(X) \subset T(X)$. Thus ***(i)*** and ***(ii)*** are completely established.

Since the pair {A, S} is weakly compatible, therefore, A and S commute at their coincidence point, i.e., $ASu = SAu$, or $Az = Sz$

similarly,

$BTv = TBv$ or $Bz = Tz$.

Now, to prove $Az = z$ by ***(2.2)***, with $\alpha = 1$

$M(Az, y_{2n+1}, a, kt) = M(Az, Bx_{2n+1}, a, kt)$
$\geq M(Tx_{2n+1}, Bx_{2n+1}, a, t) * M(Sz, Az, a, t) * M(Sz, Bx_{2n+1}, a, t) * M(Tx_{2n+1}, Az, a, t) * M(Tx_{2n+1}, Sz, a, t)$
$= M(y_{2n}, y_{2n+1}, a, t) * M(Az, Az, a, t) * M(Az, y_{2n+1}, a, t) * M(y_{2n}, Az, a, t) * M(y_{2n}, Az, a, t)$

Taking limit as $n \to \infty$

$M(Az, z, a, kt) \geq 1 * 1 * M(Az, z, a, t) * M(z, Az, a, t) * M(z, Az, a, t)$
$\geq M(Az, z, a, t)$

therefore:
by ***Lemma 1.3,*** $Az = z$

thus:

$Az = z = Sz$

similarly,

$Bz = z = Tz$.
This means that z is a common fixed point of A, B, S and T.

For uniqueness of common fixed point let $w \neq z$ be another common fixed point of A, B, S and T. Then by ***(2.2)*** with $\alpha = 1$

$M(z, w, a, kt) = M(Az, Bw, a, kt)$
$\geq M(Tw, Bw, a, t) * M(Sz, Az, a, t) * M(Sz, Bw, a, t) * M(Tw, Az, a, t) * M(Tw, Sz, a, t)$
$\geq 1 * 1 * M(z, w, a, t) * M(w, z, a, t) * M(w, z, a, t)$
$\geq M(z, w, a, t)$

therefore:

by ***Lemma 1.3,*** $z = w$. This completes the proof.

Theorem 2.2. Let (X, M, *) be a fuzzy 2-metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM-6)***. Let A, B, S, T and P be mappings from X into itself, such that:

(2.7) $P(X) \subset AB(X)$, $P(X) \subset ST(X)$

(2.8) there exists a constant $k \in (0, 1)$, such that:

$M(Px, Py, a, kt) \geq M(ABy, Py, a, t) * M(STx, Px, a, t) * M(STx, Py, a, \alpha t) * M[ABy, Px, a, (2 - \alpha) t] * M(ABy, STx, a, t)$

for all $x, y, a \in X$, $\alpha \in (0, 2)$ and $t > 0$

(2.9) If one of P(X), AB(X) or ST(X), is a complete sub-space of X

then:

(i) P and AB have a coincident point
(ii) P and ST have a coincident point

further, if

(2.10) $PB = BP$; $AB = BA$; $PT = TP$ and $ST = TS$

(2.11) the pairs {P, AB} and { P, ST} are weakly compatible

then:

(iii) A, B, S, T and P have a unique common fixed point in X

Proof. By ***(2.7)***, since $P(X) \subset AB(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$, such that, $Px_0 = ABx_1$. Since $P(X) \subset ST(X)$, for this point x_1, a point $x_2 \in X$ can be chosen, such that, $Px_1 = Sx_2$, and so on. Inductively, a sequence $\{y_n\}$ in X can be defined as:

$y_{2n} = Px_{2n} = ABx_{2n+1}$ and $y_{2n+1} = Px_{2n+1} = STx_{2n+2}$

where:

$n = 0, 1, 2, \ldots$

by ***(2.8)***, for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0,1)$, it would be:

$M(Px_{2n+2}, Px_{2n+1}, a, kt) \geq M(ABx_{2n+1}, Px_{2n+1}, a, t) * M(STx_{2n+2}, Px_{2n+2}, a, t) * M(STx_{2n+2}, Px_{2n+1}, a, \alpha t) * M[ABx_{2n+1}, Px_{2n+2}, a, (2-\alpha)t] * M(ABx_{2n+1}, STx_{2n+2}, a, t)$

$M(y_{2n+2}, y_{2n+1}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n+1}, y_{2n+1}, a, \alpha t) * M[y_{2n}, y_{2n+2}, a, (1+q)t] * M(y_{2n}, y_{2n+1}, a, t)$
$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * 1 * M[y_{2n}, y_{2n+2}, a, (1+q)t] * M(y_{2n}, y_{2n+1}, a, t)$
$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M[y_{2n}, y_{2n+2}, a, (1+q)t]$

On the lines of Sharma (2002), it is:

$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n}, y_{2n+2}, a, tq + t/2 + t/2)$
$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n}, y_{2n+2}, y_{2n+1}, qt) * M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$
$\geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t) * M(y_0, y_2, y_1, t/3q^{2n}) * M(y_0, y_1, y_1, t/3q^{2n}) * M(y_1, y_2, y_1, t/3q^{2n}) * M(y_{2n}, y_{2n+1}, a, t/2) * M(y_{2n+1}, y_{2n+2}, a, t/2)$

thus:

since $M(y_0, y_2, y_1, t/3q^{2n}) \to 1$ as $n \to \infty$

it is:

(2.12) $M(y_{2n+1}, y_{2n+2}, a, kt) \geq M(y_{2n}, y_{2n+1}, a, t) * M(y_{2n+1}, y_{2n+2}, a, t)$

similarly,

(2.13) $M(y_{2n+2}, y_{2n+3}, a, kt) \geq M(y_{2n+1}, y_{2n+2}, a, t) * M(y_{2n+2}, y_{2n+3}, a, t)$

from ***(2.12)*** and ***(2.13)*** if follows that:

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t)$

for $n = 1, 2, \ldots$, and also for positive integer n, p

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t) * M(y_{n+1}, y_{n+2}, a, t/k^P)$

thus:

since $M(y_{n+1}, y_{n+2}, a, t/k^P) \to 1$ as $p \to \infty$

$M(y_{n+1}, y_{n+2}, a, kt) \geq M(y_n, y_{n+1}, a, t)$

therefore:

by ***Lemma 1.2***, $\{y_n\}$ is a Cauchy sequence in X.

Now, suppose ST(X) is complete
note that the subsequence $\{y_{2n+1}\}$ is contained in ST(X) and has a limit in ST(X) called z
Let $u = (ST)^{-1}z$, then $STu = z$.

Applying the fact that the sub-sequence $\{y_{2n}\}$ also converges to z

by ***(2.8)*** with $\alpha = 1$, it would be:

$M(Pu, y_{2n+1}, a, kt) = M(Pu, Px_{2n+1}, a, kt)$
$\geq M(ABx_{2n+1}, Px_{2n+1}, a, t) * M(STu, Pu, a, t) * M(STu, Px_{2n+1}, a, t) * M(ABx_{2n+1}, Pu, a, t) * M(ABx_{2n+1}, STu, a, t)$
$= M(y_{2n}, y_{2n+1}, a, t) * M(z, Pu, a, t) * M(z, y_{2n+1}, a, t) * M(y_{2n}, Pu, a, t) * M(y_{2n}, z, a, t)$

which implies that as $n \to \infty$

$M(Pu, z, a, kt) \geq 1 * M(z, Pu, a, t) * 1 * M(z, Pu, a, t) * 1$
$\geq M(Pu, z, a, t)$

therefore:

by ***Lemma 1.3***, $Pu = z$. Since $STu = z$, $Pu = z = STu$, i.e., u is coincidence point of P and ST. This proves ***(i)***.

Since $P(X) \subset AB(X)$, $Pu = z$ implies that $z \in AB(X)$.

Let $v = (AB)^{-1}z$, then $ABv = z$. It can easily be verified by using similar argument of the previous part of the proof that $Pv = z$. If it is assumed that AB(X) is complete, then argument analogous to the previous completeness argument establishes ***(i)*** and ***(ii)***.

The remaining one case pertains, essentially, to the previous cases indeed. If P(X) is complete, then by ***(2.7)***, $z \in P(X) \subset ST(X)$, or $z \in P(X) \subset AB(X)$. Thus ***(i)*** and ***(ii)*** are completely established. Since, the pair {P, ST} is weakly compatible, therefore, P and ST commute at their coincident point, i.e., $P(STu) = (ST)Pu$, or $Pz = STz$. Similarly, $P(ABv) = (AB)Pv$ or $Pz = ABz$.

Now, to prove that $Pz = z$, by ***(2.8)*** with $\alpha = 1$, it would be:

$M(Pz, y_{2n+1}, a, kt) = M(Pz, Px_{2n+1}, a, kt)$
$\geq M(ABx_{2n+1}, Px_{2n+1}, a, t) * M(STz, Pz, a, t) * M(STz, Px_{2n+1}, a, t) * M(ABx_{2n+1}, Pz, a, t) * M(ABx_{2n+1}, STz, a, t)$
$= M(y_{2n}, y_{2n+1}, a, t) * M(Pz, Pz, a, t) * M(Pz, y_{2n+1}, a, t) * M(y_{2n}, Pz, a, t) * M(y_{2n}, Pz, a, t)$

Taking the limit $n \to \infty$, it is:

$M(Pz, z, a, kt) \geq 1 * 1 * M(Pz, z, a, t) * M(z, Pz, a, t) * M(z, Pz, a, t)$
$\geq M(Pz, z, a, t)$

therefore:

by ***Lemma 1.3***, $Pz = z$, thus $ABz = z = Pz = STz$.

Now, it will be shown that $Bz = z$. In fact by ***(2.8)*** with $\alpha = 1$ and ***(2.10)*** it is:

$M(z, Bz, a, kt), = M(Pz, BPz, a, kt) = M(Pz, PBz, a, kt)$
$\geq M[AB(Bz), PBz, a, t] * M[STz, Pz, a, t] * M[STz, PBz, a, t] * M[AB(Bz), Pz, a, t] * M[AB(Bz), STz, a, t]$
$= 1 * 1 * M(z, Bz, a, t) * M(Bz, z, a, t) * M(Bz, z, a, t)$
$\geq M(z, Bz, a, t)$

which implies, by ***Lemma 1.3***, that $Bz = z$. Since $ABz = z$, therefore, $Az = z$. Finally, $Tz = z$. Indeed, by ***(2.8)*** with $\alpha = 1$ and ***(2.10)***

$M(Tz, z, a, kt) = M(TPz, Pz, a, kt) = M(PTz, Pz, a, kt)$

$\geq$ M[ABz, Pz, a, t] * M[ST(Tz), PTz, a,t] * M[ST(Tz), Pz, a,t] * M[ABz, P(Tz), a, t] * M[ABz, ST(Tz), a, t]
= 1 * 1 * M(Tz, z, a, t) * M(z, Tz, a, t) * M(z, Tz, a, t)
$\geq$ M(Tz, z, a, t)

which implies, by ***Lemma 1.3***, that Tz = z. Since STz = z, z = STz = Sz. Therefore, by combining the above results, Az = Bz = Sz = Tz = Pz = z, i. e., z is a common fixed point of A, B, S, T and P.

For uniqueness of common fixed point, let w $\neq$ z be another common fixed point of A, B, S, T and P.

then:

by ***(2.8)*** with $\alpha = 1$, it would be:

M(z, w, a, kt) $\geq$ M(ABw, Pw, a, t) * M(STz, Pz, a, t) * M(STz, Pw, a, α t) * M(ABw, Pz, a, t) * M(ABw, STz, a, t)
$\geq$ 1 * 1 * M(z, w, a, t) * M(w, z, a, t) * M(w, z, a, t)
$\geq$ M(z, w, a, t)

therefore:

by ***Lemma 1.3***, we have z = w. This completes the proof.

Theorem 2.3. Let (X, M, *) be a fuzzy-2 metric space with t* t $\geq$ t for all t $\in$ [0, 1] and the condition ***(FM-6)***. Let A, B, S, T, P and Q be mappings from X into itself, such that:

(2.14) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$

(2.15) there exists a constant $k \in (0, 1)$, such that:

M(Px, Qy, a, kt) $\geq$ M(ABy, Qy, a, t) * M(STx, Px, a, t) * M(STx, Qy, a, α t) * M[ABy, Px, a, $(2-\alpha)$t] * M(ABy, STx, a, t)

for all x, y, a $\in$ X, $\alpha \in (0, 2)$ and $t > 0$

(2.16) If one P(X), Q(X), AB(X) or ST(X) is complete subspace of X

then:

(i) P and ST have a coincident point
(ii) Q and AB have a coincident point

further, if

(2.17) AB = BA, QB = BQ , PT = TP and ST = TS

(2.18) the pairs {Q, AB} and {P, ST} are weakly compatible

then:

A, B, S, T, P and Q have a unique common fixed point in X.

Proof. By ***(2.14)***, since $P(X) \subset AB(X)$, for any point $x_0 \in X$, there exists a point $x_1 \in X$, such that $Px_0 = ABx_1$. Since $Q(X) \subset ST(X)$, for this point x_1, a point $x_2 \in x$ can be chosen, such that:

$Qx_1 = STx_2$, and so on

inductively, a sequence $\{y_n\}$ in X can be defined as:

$y_{2n} = Px_{2n} = ABx_{2n+1}$, and
$y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$

where:

$n = 0, 1, 2, \ldots$

For all t > 0 and $\alpha = 1 - q$, with $q \in (0,1)$. As proved in **Theorems *(2.1)*** and **2.2**, it can be proved that $\{y_n\}$ is a Cauchy sequence in X. Now, suppose ST(X) is complete; note that ST(X) contains the sub-sequence $\{y_{2n+1}\}$and has a limit in ST(X), called z. Let $u = (ST)^{-1}z$, then STu = z. Applying the fact that the sub-sequence $(y_{2n}\}$ also converges to z

by ***(2.15)*** with $\alpha = 1$, it would be:

M(Pu, Qx_{2n+1}, a, kt) $\geq$ M(ABx_{2n+1}, Qx_{2n+1}, a, t) * M(STu, Pu, a, t) * M(STu, Qx_{2n+1}, a, t) * M(ABx_{2n+1}, Pu, a, t) * M(ABx_{2n+1}, STu, a, t)
= M(y_{2n}, y_{2n+1}, a, t) * M(STu, Pu, a, t) * M(STu, y_{2n+1}, a, t) * M(y_{2n}, Pu, a, t) * M(y_{2n}, STu, a, t)

which implies that as $n \to \infty$

M(Pu, z, a, kt) $\geq$ M(Pu, z, a, t)

therefore:

by ***Lemma 1.3***, Pu = z. Since STu = z, thus Pu = z = STu, i.e., u is a coincidence point of P and ST. This proves ***(i)***. Since $P(X) \subset AB(X)$ and Pu = z implies that $z \in AB(X)$.

Let $v = (AB)^{-1}$ z, then ABv = z

by ***(2.15)*** with $\alpha = 1$, it would be:

M(Pz, y_{2n+1}, a, kt) = M(Pz, Qx_{2n+1}, a, kt)
$\geq$ M(ABx_{2n+1}, Qx_{2n+1}, a, t) * M(STz, Pz, a, t) * M(STz, Qx_{2n+1}, a, t) * M(ABx_{2n+1}, Pz, a, t) * M(ABx_{2n+1}, STz, a, t)
$\geq$ M(y_{2n}, y_{2n+1}, a, t) * M(STz, Pz, a, t) * M(STz, y_{2n+1}, a, t) * M(y_{2n}, Pz, a, t) * M(y_{2n}, STz, a, t)

Taking the limit as $n \to \infty$, it is:

M(Pz, z, a, kt) $\geq$ M(Pz, z, a, t)

therefore:

by ***Lemma 1.3***, we have Pz = z = STz.

Now, it shows that Qz = z. In fact by *(2.15)* with $\alpha = 1$ and ***(2.17),*** it would be:

M(y_{2n}, Qz , a, kt) = M(Px_{2n}, Qz, a, kt)
$\geq$ M(ABz, Qz, , a, t) * M(STx_{2n} , Px_{2n}, a, t) * M(STx_{2n}, Qz, a, t) * M(ABz, Px_{2n}, a, t) * M(ABz, STx_{2n}, a, t)

Taking the limit $n \to \infty$

M(z, Qz, a, t) $\geq$ M(Qz, z, a, t)

therefore:

by ***Lemma 1.3,*** $Qz = z = ABz$. Thus, $Pz = Qz = ABz = STz = z$.

By putting $x = z$ and $y = Bz$, with $\alpha = 1$ in ***(2.15)***, using ***(2.17)*** and ***Lemma 1.3***, it is easy to see that $Bz = z$. Since $ABz = z$, therefore, $Az = z$

similarly,

by putting $x = Tz$ and $y = z$, with $\alpha = 1$ in ***(2.15)***, using ***(2.17)*** and ***Lemma 1.3***, it is easy to prove that $Tz = z$. Since $STz = z$, $Sz = z$. Therefore, by combining the above results, it would be:

$$Az = Bz = Sz = Tz = Qz = z$$

which mean that z is the common fixed point of A, B, S, T, P and Q. Thus, it is easy to prove uniqueness.

Theorem 2.4. Let $(X, M, *)$ be a fuzzy metric 2-space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM-6)***. Let A, B, S, T and $\{P_i\}_{i \in I}$ be mappings from x into itself:

(2.19) $\cup_{i \in I} P_I(X) \subset AB(X)$, $\cup_{i \in I} P_i(X) \subset ST(X)$, where I is an index set

(2.20) there exists a constant $k \in (0, 1)$, such that:

$M(P_i x, P_i y, a, kt) \geq M(ABy, P_i y, a, t) * M(STx, P_i x, a, t) * M(STx, P_i y, a, \alpha t) * M[ABy, P_i x, a, (2-\alpha)t] * M(ABy, STx, a, t)$

for all $x, y, a \in X$, $\alpha \in (0, 2)$, $i \in I$ and $t > 0$

(2.21) If one of $AB(X)$, or $ST(X)$, or $P_i(X)$ $(i \in I)$ is a complete subspace of X, then:

(i) for all $i \in I$, P_i and AB have a coincidence point
(ii) for all $i \in I$, P_i and ST have a coincidence point

further, if

(2.22) for all $i \in I$, $P_i B = BP_i$ $AB = BA$; $P_i T = TP_i$ and $ST = TS$

(2.23) for all $i \in I$, the pairs $\{P_i, AB\}$ and $\{P_i, ST\}$ are weakly compatible, then

(iii) A, B, S, T and $\{P_i\}_{i \in I}$ have a unique common fixed point in X

if we put $B = T = I_x$ (the identity map on X) in **Theorem 2.2,** the following result:

Corollary 2.1. Let $(X, M, *)$ be a fuzzy 2-metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM-6)***. Let A, S and P be mappings from X into itself, such that:

(2.24) $P(X) \subset A(X)$, $P(X) \subset S(X)$

(2.25) there exists a constant $k \in (0, 1)$, such that:

$M(Px, Py, a, kt) \geq M(Ay, Py, a, t) * M(Sx, Py, a, \alpha t) * M[Ay, Py, a, (2 - \alpha) t] * M(Ay, Sx, a, t)$

for all $x, y, a \in X$, $\alpha \in (0, 2)$ and $t > 0$

(2.26) If one of P(X), or S(X) is a complete sub-space of X, then:

(i) P and A have a coincident point
(ii) P and S have a coincident point

further, if

(2.27) The pair $\{P, A\}$ and $\{P, S\}$ are weakly compatible, then:

(iii) A, S and P have a unique common fixed point in X

if, $A = B = S = T = I_x$ (the identity mapping on X) in **Theorem 2.2,** the following results are acquired:

Corollary 2.2. Let $(X, M, *)$ be a fuzzy metric 2-space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM-6)***. Let P be mappings from X itself, such that:

(2.28) there exists a constant $k \in (0, 1)$, such that:

$M(Px, Py, a, kt) \geq M(y, Py, a, t) * M(x, Px, a, t) * M(x, Py, a, \alpha t) * M[y, Px, a, (2 - \alpha) t] * M(y, x, a, t)$

for all $x, y, a \in X$, $\alpha \in (0, 2)$ and $t > 0$

if P(X) is a complete sub-space of X, then P has a unique common fixed point in X

by using **Theorem 2.1**, the following results are acquired:

Theorem 2.5. Let $(X, M, *)$ be a fuzzy 2-metric space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM-6)***. Let A, B and T be mappings from X into itself, such that:

(2.29) $A(X) \cup B(X) \subset T(X)$

(2.30) there exists a constant $k \in (0, 1)$, such that:

$M(Ax, By, a, kt) \geq M(Ty, By, a, t) * M(Tx, Ax, a, t) * M(Tx, By, a, \alpha t) * M[Ty, Ax, a, (2 - \alpha) t] * M(Ty, Tx, a, t)$

for all $x, y, a \in x$, $\alpha \in (0, 2)$ and $t > 0$

(2.31) One of A(X), B(X) or T(X) is a complete sub-space of X

then:

A, B, T have a coincidence point

thus:

Theorem 2.1, for sequence of mappings, is established in the following manner:

Theorem 2.6. let $(X, M, *)$ be a fuzzy metric 2-space with $t * t \geq t$ for all $t \in [0, 1]$ and the condition ***(FM- 6)***. Let $S, T, A_i : X \to X$, $i = 0, 1, 2, \ldots$, such that:

(2.32) $A_0(X) \subset T(X)$, $A_i(X) \subset S(X)$, $i \in N$

(2.33) there exists a constant $k \in (0, 1)$, such that:

$M(A_0 x, A_i y, a, kt) \geq M(Ty, A_i y, a, t) * M(Sx, A_0 x, a, t) * M(Sx, A_i y, a, \alpha t) * M[Ty, A_0 x, a, (2\text{-}\alpha) t] * M(Ty, Sx, a, t)$

for all x, y, $a \in X$, $\alpha \in (0, 2)$ and $t > 0$

(2.34) the pairs $\{A_0, S\}$ and $\{A_i, T\}$ ($i \in N$) are weakly compatible

(2.35) if one of S(X), T(X) or $A_0(X)$ is a complete sub-space of X, or alternatively A_i, $i \in N$ are complete sub-space of X

then:

S, T and A_i, $i = 0, 1, 2, \ldots$ have a unique common fixed point.

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