Pak. J. Sci. Ind. Res. 2005 **48**(3) 149-153

A Class of Algorithms for Zeros of Polynomials

Stephen Ehidiamhen Uwamusi
Department of Mathematics, Faculty of Science, University of Benin, Edo State, Nigeria

(received October 17, 2003; revised October 30, 2004; accepted January 4, 2005)

Abstract. A class of polytype algorithms for the determination of zeros of polynomial equation of single variable was constructed from Jarratt's asymptotic error constant using the well-known Chevbyshev's comparison series. The method was synthesized from the Taylor series iterative method. The climax of the findings was that the proposed formula can effectively model the determination of zeros of polynomial equations of the type often encountered in the Givens orthogonal matrix transformation process.

Keywords: polynomial zeros, Jarratt's asymptotic error constant, Taylor series, Laguerre iterative method, Givens orthogonal matrix transformation

Introduction

The aim was to solve a non-linear operator equation, $p(z) = 0$, by an iteration that builds up a sequence of approximation $\{z_i\}$ iteratively, where it was assumed that p is sufficiently smooth in its domain of definition. Laguerre's method is an iterative operator which uses non-linear information, the degree of the polynomial whose root was being sought. Thus, there exist two widely circulated iterative operators for approximating roots of non-linear equations, namely, the stationary and non-stationary methods, respectively.

The stationary iteration is one-point iteration operator without memory. This means that it is independent on previously calculated values, unlike the case of the Secant method, as well as the Steffensen iteration, which are the members of non-stationary iteration and are regarded as multipoint stationary iteration operators with memory. To motivate further interest in this, the same reasoning as of Wasilkowski (1980) was applied in the case of non-stationary iterative operator. This paper, as a result, develops a class of polytype algorithms derived from the Taylor series iterative formula using Jarratt's asymptotic error constant (Jarratt, 1968) for the determination of zeros of polynomial equation obtained from the Givens orthogonal matrix plane rotation (Wilkinson, 1965). For easy orientation, the Taylor series expansion for $p(z)$, about the $z = z_i$, has been defined as follows:

$$p(z) = p(z_i) + p'(z_i)\,(z-z_i) + p''\,(z_i)\,\frac{(z-z_i)^2}{2!} + \sum_{j=3}^{\infty} p^{(j)}\,\frac{(z-z_i)^j}{j!} \quad (1.1)$$

If it is set as $p(z) = 0$ and take cognizance of the fact that $z_i^{(k)} \in z_i\ (k = 0, 1, ...)$, then the Taylor polynomial would be in the form:

$$0 = p(z_i^{(k)}) + p'\,(z_i^{(k)})(z-z_i^{(k)}) + p''\,(z_i^{(k)})\,\frac{(z-z_i^{(k)})^2}{2!} + p'''\,(z_i^{(k)})\,\frac{(z-z_i^{(k)})^3}{3!} \quad (1.2)$$

because:

$z_i - z_i^{(k)} \in z_i^{(k+1)} - z_i^{(k)}$, so that in the limit $\zeta \in z_i$

where:

$\zeta_i\,(i = 1, 2....n)$ in the case of simple zeros, approximate sufficiently close to the zeros of $p(z_i)$, then it would be:

$$\zeta_i = z_i^{(k)} - \frac{1}{p'(z_i^{(k)})}\left[p(z_i^{(k)}) + \sum_{m=2}^{i+1} p^{(m)}(z_i^{(k)})\ \mathrm{x}\ \frac{(z_i^{(k+1)} - z_i^{(k)})^m}{m!} + 0(.)^{i+2}\right] \cap z_i^{(k)} \quad (1.3)$$

In view of method (1.3), the nested sequence would be:

$z_i^{(k)} \supset z_i^{(k+1)} \supset ... \supset z_i^{(m)}$

Neglecting the terms higher than second order derivatives in method (1.3), a convergent iterative method with non-intersection property can be obtained:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{1}{p'\,(z_i^{(k)})}\left[p\,(z_i^{(k)}) + \frac{p''\,(z_i^{(k)})}{2}(z_i^{(k+1)} - z_i^{(k)})^2\right] \cap z_i^{(k)} \quad (1.4)$$

where:

$$z_i^{(k+1)} = \left[z_i^{(k)} - \frac{p\,(z_i^{(k)})}{p'\,(z_i^{(k)})}\right] \cap\ z_i^{(k)} \quad (1.5)$$

The foregoing is a preliminary discussion for deriving non-stationary iterative operator for the solution of zeros of poly-

E-mail: uwamusi2000@yahoo.com

nomial. The sample problem is obtained from Givens matrix orthogonal similarity transformation applied on a certain matrix (Jackson, 1975).

Results and Discussion

In the sequel, the following is defined as an iterative sequence:

$$z_i^{(k+1)} = \Phi\,(z_i^{(k)}),\ (k = 0, 1, ...) \tag{2.1}$$

where:

$\Phi(z) \rightarrow z_i - \delta(z_i)$ = rational map that preserves self-mapping

A multipoint iteration of Jarratt (1968) is defined in the form:

$$\Phi(z) = \frac{6\,p(z)}{w_1(z) + w_2(z) + 4w_3(z)} \tag{2.2}$$

where:

$$w_1 = p'$$

$$w_2 = p'\left[z - \frac{p}{w_1}\right]$$

$$w_3 = p'\left[z - \frac{1}{8}\frac{p}{w_1} - \frac{3}{8}\frac{p}{w_2}\right]$$

Now, for a given expression $p(z+h)$, the Taylor series expansion may be defined in the form:

$$p(z+h) = C_0 + C_1h + C_2h^2 + ... \tag{2.3}$$

Differentiating (2.3), it would be:

$$p'(z+h) = C_1 + 2C_2h + 3C_2h^2 + ...+.... \tag{2.4}$$

Now, using the idea in (2.4), $p[z-U(z)]$ is expanded in the form (where it set):

$$U(z) = \left[\frac{p(z)}{p'(z)}\right]$$

$$p'[z-U(z)] = p'(z)\,[1-2A_2U + 3A_3U^2 - 4A_4U^3 + 5A_5U^4 + 0(U)^5] \tag{2.5}$$

and $A_j = \dfrac{p^{(j)}(z)}{j!\,p'(z)}$

Associate to $\Phi(z)$, is the Chevbyshev's comparison series method:

$$E_p(z) = z - \sum_{t=1}^{p-1} Y_t U^t \tag{2.6}$$

with $Y_1(z) = 1$ and the Y_t satisfies the difference differential equation:

$$tY_t(z) - 2(t-1)A_2(z)\,Y_{t-1}(z) + Y'_{t-1}(z) = 0 \tag{2.7}$$

The expression may be simplified in the form as found in Jarratt (1968):

$$\frac{p(z)}{p'[z - U(z)]} = U + 2A_2U^2 + (4A_2^{\,2} - 3A_3)\,U^3 + (4A_4 - 12A_2A_3 + 8A_2^{\,2})\,U^4 + 0(U)^5 \tag{2.8}$$

and

$$p'\left[z - \frac{1}{8}\frac{p}{w_1} - \frac{3}{8}\frac{p}{w_2}\right] = p'(z)\left[1 - A_2U - \frac{3}{4}(2A_2^{\,2} - A_3)\right]U^2 - \left[(3A_2^{\,3} - \frac{0}{2}A_2A_3 + \frac{1}{2}A_4\right]U^3 - \left[6A_2^{\,4} - \frac{243}{10}A_2^{\,2}A_3 + \frac{27}{8}A_3^{\,2} + \frac{21}{4}A_2A_4 - \frac{5}{16}A_5\right]U^5 + 0(U)^5 \tag{2.9}$$

using:

$$\Phi(z) = \frac{p(z)}{\sum_{t=1}^{5} q_i U^{i-1}} \tag{2.10}$$

where:

$q_1 = 1;\ q_2 = -A_2;\ q_3 = -A_2^{\,2} + A_3;\ q_4 = -2A_2^{\,3} + 3A_2A_3 + A_4;$

$$q_5 = -4A_2^{\,4} + \frac{81}{8}A_2^{\,2}A_3 - \frac{0}{4}A_3^{\,2} - \frac{7}{3}A_2A_4 + \frac{25}{24}A_5$$

then Jarratt (1968) obtained the following expression using binomial series:

$$\Phi(z) = z - U + q_2U^2 - (q_2^{\,2} - q_3)\,U^3 + (q_2^{\,3} - 2q_2q_3 + q_4)\,U^4 - (q_2^{\,4} - 3q_2^{\,2}q_3 + q_3^{\,2} + 2q_2q_4 - q_5)\,U^5 + 0(U)^6 \tag{2.11}$$

To obtain the methods, the following manner was proceeded, which is rather comic analogously:

truncating $\Phi(z) = z - U$ and substituting for $z_i^{(k+1)}$ in method (1.3), the method (1.4) would yield, if terms higher than $0(z_i^{(k+1)} - z_i^{(k)})^2$ are neglected.

Similarly, writing:

$$\Phi(z_i^{(k)}) = z_i^{(k)} - U + q_2U^2$$

and substituting the expression for $z_i^{(k+1)}$ into method (1.4), to a convergent iterative method in the following form would result:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{1}{p'(z_i^{(k)})}\left[p(z_i^{(k)}) + \frac{p''(z_i^{(k)})}{2}(z_i^{(k+1)} - z_i^{(k)})^2\right] \cap z_i^{(k)} \tag{2.12}$$

$(k = 0, 1, ...)$

where:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p(z_i^{(k)})}{p'(z_i^{(k)})}\left[1 + \frac{p''(z_i^{(k)})\,p(z_i^{(k)})}{p'(z_i^{(k)})^2}\right] \tag{2.13}$$

If $\Phi(z)$ is truncated in (2.10) after powers higher than U^3, and written:

$p(z) = p,\ p'(z) = p',\ p''(z) = p''$, etc., it is then:

$$\Phi(z) = z - U(q_2 U - q_2^2 U^2 + q_3 U^2) =$$

$$z - \frac{p}{p'}\left[-A_2\frac{p}{p'} - (A_2)^2\,\frac{p^2}{(p')^2} + (-A_2^2 + A_3)\,\frac{p^2}{(p)^2}\right] \tag{2.14}$$

Thus, substituting the right hand side of the last expression into (1.4) an iterative formula of at least fourth order was obtained. However, this entailed evaluation of p'''. Hence, this method would not be considered further.

It is remarked that the iterative formula (2.13) can be obtained in a different way using a modified one point iterative formula as considered by Uwamusi (2004), defined in the sense of Petkovic and Trickovic (1995) that if:

$$z_i^{(k+1)} = g\,(z_i^{(k)})\ (k = 0, 1, ...) \tag{2.15}$$

and

$g(z) \to z_i - \Phi(z_i)$ as a one point method, such that for $m \geq 2$, an iterative method of order m could be modified in the sense of Petkovic and Trickovic (1995) such that:

$$z_i^{(k+1)} = z_i^{(k)} - \Phi\,(z_i^{(k)})\left[1 + \frac{1}{m}\,g'\,(z_i^{(k)})\right] \tag{2.16}$$

Other approaches for obtaining (2.13) may be through the Taylor polynomial of degree 2 by substituting $(z_i^{(k+1)} - z_i^{(k)})^2$ with Halley's correction

$$H(z_i^{(k)}) = \frac{p(z_i^{(k)})}{p'(z_i^{(k)})}\left[1 + \frac{p\,(z_i^{(k)})\,p''(z_i^{(k)})}{2\,p'(z_i^{(k)})}\right] \tag{2.17}$$

and taking a limit of the term appearing in the denominator part of the resulting expression, the substitution by Newton's correction in the Taylor polynomial of degree 2 was also adequate in obtaining method (2.13).

The immediate consideration was when to accept the computed result from (2.12) as being good enough as the desired zero of $p(z)$. In line with Shampine (1981) there were four ways in which the computation of a new step may fail. These are as follows:

(i) the iteration for solving (2.13) is not contracting from $z_i^{(0)}$;

(ii) the iteration for solving (2.13) is contracting from $z_i^{(0)}$, but is too slow;

(iii) the iteration for solving (2.13) is contracting from $z_i^{(0)}$ at an acceptable rate, but $z_i^{(0)}$ is too far from the solution z_i^* to get convergence in allowed number of iterations; and

(iv) the solution of (2.13) is computed but it satisfies the condition

$$C\,\|\,z_i^* - z_i^{(0)}\,\| > \varepsilon \tag{2.18}$$

for a suitable constant C and a norm. Details of such a procedure for accepting or rejecting a computed result from method (2.13) were well analysed and discussed by Shampine (1981).

Numerical example. Consider the matrix eigenvalue problem taken form Jackson (1975) given as problem 1:

$$A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{bmatrix}$$

On applying the Givens orthogonal plane rotation to A in the form as below:

$$A^{(k+1)} = Q^k\,A^{(k)}\,Q^{(k)} \tag{3.1}$$

$(k = 0, 1, ...)$

where:

$$Q = \begin{array}{cc} \left[\begin{array}{cccccccc} 1 & & & & & & & \underline{0} \\ & \ddots & & & & & & \\ & & C & & & -S & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & \ddots & & & \\ & & S & & & C & & \\ & & & & & & 1 & \\ & & & & & & \ddots & \\ \underline{0} & & & & & & & 1 \end{array}\right] & \begin{array}{c} \\ \\ -k \\ \\ \\ \\ -j \\ \\ \\ \\ \end{array} \\ \begin{array}{cccccccc} & & \overset{1}{k} & & & \overset{1}{j} & & \end{array} & \end{array}$$

The (a_{kj}) denotes the matrix elements in k, j position.

Because Q is a transformation group that preserves length, it is easy to observe that a tridiagonal matrix of the type found in Golub and Van-Loan (1983) would be obtained:

$T = Q^{-1}\ AQ$

The quantities C and S obtained accordingly would be:

$C = y^{-1} k,\ S = y^{-1} j$

where:

$y = (|k|^2 + |j|^2)^{½}$

and that:

$C^2 + S^2 = 1$

$-S_k + C_j = 0$

thus, the problem number 1 becomes:

$$A^{(1)} = \begin{bmatrix} 2 & 3.16227766 & 0 & 4 \\ 3.16227766 & 5.699999999 & 1.9 & -3.16227766 \\ 0 & 1.9 & -2.7 & -5.375872022 \\ 4 & -0.316227766 & -5.375872022 & -1 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 2 & 5.099019508 & 0 & 0 \\ 5.99019508 & 1.269230765 & -3.038850996 & 3.186602865 \\ 0 & -3.038850996 & -2.7 & -4.824456918 \\ 0 & -3.186602865 & -4.824456918 & -3.430769226 \end{bmatrix}$$

$$A^{(3)} = \begin{bmatrix} 2 & 5.099019508 & 0 & 0 \\ 5.99019508 & 1.269230765 & -4.403300258 & 0 \\ 0 & -4.403300258 & -4.308217564 & 3.290806892 \\ 0 & 0 & 3.290806892 & 5.038986789 \end{bmatrix}$$

Application of Sturm's sequence to $A^{(3)}$ will yield:

$p_4(z) = z^4 - 3.999999999\, z^3 - 72.99999972\, z^2 + 259.9999983z + 505.7006909 = 0$

The results for $p_4(z)$ are presented by using the well-known Laguerre's method (Wilkinson, 1965) as a case study in comparison with (2.12). The Laguerre's method is as follows:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{np(z_i^{(k)})}{p'(z_i^{(k)}) \pm [(n-1)^2\, p'(z_i^{(k)})^2 - n(n-1)\, p(z_i^{(k)})p''(z_i^{(k)})]^{½}} \tag{3.1}$$

The following results are presented in Table 1 and 2.

Conclusion

From Table 1 and 2, it is self-verifying that the proposed class of methods endorsed fixed point principle and convergence to a desired zero was monotonic. Using the fact that the sum of the roots of $p(z)$ in the example was 3.999999999, which was again equal to the trace of $A^{(0)}$ and hence $A^{(3)}$, it maybe concluded that the method (2.12) was not only stable and easy to use but should also be recommended for a general purposed computational utility for finding zeros of algebraic equations. The convergence to the root of the method (2.12) depended materially on the term:

$$\frac{1}{2}\, p''\,(z_i^{(k)})\,(z_i^{(k+1)} - z_i^{(k)})^2$$

Table 1. Results for method (3.1)

Number of iterations	$z_1^{(k)}$	$z_2^{(k)}$	$z_3^{(k)}$	$z_4^{(k)}$
0	12	0	8.075	5.5
1	8.328488792	-1.364498471	-8.071255234	5.405953266
2	8.0976382	-1.431151518	-8.072165782	5.405922779
3	8.097402142	-1.431159324	-8.072165783	5.405922779
4	8.097402142	-1.43115914	-8.072165783	
5		-1.43115914		

Table 2. Results for method (2.12)

Number of iterations	$z_1^{(k)}$	$z_2^{(k)}$	$z_3^{(k)}$	$z_4^{(k)}$
0	9	0	-8	5.5
1	8.135126491	-1.726167961	-8.072177917	5.405981036
2	8.097399609	-1.431797537	-8.072165782	5.405922779
3	8.097402144	-1.431159163	-8.072165782	5.405922779
4	8.097402142	-1.43115914		

Table 3. Results for method (2.13) as predictor to method (2.12)

Number of iterations	$z_1^{(k)}$	$z_2^{(k)}$	$z_3^{(k)}$	$z_4^{(k)}$
0	9	0	-8	5.5
1	8.206508596	-0.882842732	-8.072109813	5.40603511
2	8.097427721	-1.432072424	-8.072165782	5.405922779
3	8.097402144	-1.431159163	-8.072165782	5.405922779
4	8.097402142	-1.43115914		

as this approaches a zero number at a very fast rate as the sequence of iterates $\{z_i\} \to \propto$. As it were, Laguerre's method involved the computation of square root, which may be quite expensive in terms of computation time. Moreover, Laguerre's method, like any other square root methods, may branch off into complex plane even if the roots of polynomial p were real, especially when $p'^2 > pp''$. Again, from the computational experience with the method (2.12), it may be seen that the numerical values obtained from method (2.12) usually coincide with (2.13) after a few steps of iteration were started (see Table 3 for more details).

References

Golub, G.H., Van-Loan, C.F. 1983. *Matrix Computations*, pp. 43-47, pp. 267-307, Johns Hopkins University Press, Baltimore, Maryland, USA.

Jackson, L.W. 1975. Interval arithmetic error-bounding algorithms. *SIAM J. Numer. Analysis* **12:** 223-238.

Jarratt, P. 1968. The use of comparison series in analyzing iteration functions. *The Computer J.* **11:** 314-316.

Petkovic, M.S., Trickovic, S. 1995. On zero finding method of the fourth order. *J. Computational Appl. Math*. **64:** 291-294.

Shampine, L.F. 1981. Efficient use of implicit formulas with predictor-corrector error estimate. *J. Computational Appl. Math.* **7:** 33-35.

Uwamusi, S.E. 2004. A family of iteration formulas for the determination of the zeros of a polynomial. *Pak. J. Sci. Ind. Res.* **47:** 9 -12.

Wasilkowski, G.W. 1980. Can any stationary iteration using linear information be globally convergent? *J. Assoc. Computing Machinery* **27:** 263-269.

Wilkinson, J.H. 1965. *The Algebraic Eigenvalue Problem*, pp. 282-307, Oxford University Press, Ely House, London, U.K.