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A Family of Iteration Formulas for the Determination of the Zeros of a Polynomial

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We present a class of globally monotonically convergent iterative methods for the determination of zeros of a polynomial. The proposed method uses the well-known Newton's second order method as a basic ingredient to generate this class of methods, following the approach of Petkovic and Trickovic as supported by Cauchy Schwartz inequality from which come in hand three methods of fourth order. The obtained methods can be used to provide tight inclusion conditioning bounds separating the sought zeros. This means that they always provide good numerical approximations within the theoretical conditioning bounds. It is found that one of the fourth order methods so obtained competes most favourably with any known methods for finding zeros of a polynomial.

Key words: Newton's methods, Halley's methods, Chevbyshev's method, Polynomial zeros.

Introduction

When a zero seeking algorithm has local convergency properties, the iteration if it converges will display a doubling phenomenon at each iteration. The point at which an iterative sequence starts experiencing a contraction immediately an iterative process is began called a point of attraction or else we say that the iterative sequence diverges which is known as point of repulsion. Of major importance in this paper is the derivation of two new methods through the technique of Petkovic and Trickovic (1995) as supported by the concept of Cauchy - Schwartz inequality $\left(\Sigma \frac{1}{Z - Z_i}\right)^2 \leq n \, \Sigma \left(\frac{1}{Z - Z_i}\right)^2$ where, $Z_1, Z_2 \ldots Z_n$ are the roots of polynomial of degree n for the case of real simple zeros, given by: $P(Z) = \sum_{j=0} a_j Z^j, a_j \in ⊄$.

It is found that one of the two new methods obtained competes favourably with the popular Chevbyshev third order method for the extraction of zeros of a polynomial.

In a note of passing we shortend notation by omitting the argument Z from function such as P(Z), and write simply P. We also denote the first three derivatives of a function P by P′,P″ and P‴, respectively. It is also supposed that the functions discussed in this paper has what ever number of C^{n+1} necessary on the interval.

Experimental

Derivation of the class of methods. The class of methods under consideration is a particular case of a general one point iterative formula considered in Petkovic and Trickovic (1995) as well as Milovanovic (1974) which is as follows, we define:

$$Z_i^{(k+1)} = g'(Z^{(k)}), (k = 0, 1 \ldots) \qquad (2.1)$$

where; $g(Z) \rightarrow Z_i - \Phi(Z_i)$ as a one point method.
Observe that $\Phi(Z_i)$ is a rational map which converges such that: $Z^{k+1} - Z^k \rightarrow 0$.

For $m \geq 2$ an interative method of order m could be modified (see, Petkovic and Trickovic (1995) for more details) such that:

$$Z_i^{(k+1)} = Z_i^{(k)} - \left(\Phi(Z^{(k)}) \left(1 + \frac{1}{m} g'(Z_i^{(k)})\right) \right) \qquad (2.2)$$

In this paper, starting from Newton's Second order method we shall generate such higher methods which are structurally similar to those obtained by Jarrat (1968). Method (2.2) is a major plank in which our derivation heavily leans on.
Define Newton's formula as:

$$Z_i^{(k+1)} = Z_i^{(k)} - \frac{p'}{p'} \; (k = 0, 1 \ldots.) \qquad (2.3)$$

In view of (2.1) and (2.2) we write (2.3) in the form:

$$\hat{Z}_i = Z_i - \frac{p}{p'} \left(1 + \frac{1}{2} \; g'(Z)\right)$$

$$= Z_i - \frac{p}{p'} \left(1 + \frac{1}{2} \frac{d}{dz} \left(Z_i - \frac{p}{p'}\right)\right)$$

$$= Z_i - \frac{p}{p'} \left(1 + \frac{1}{2} \frac{(p'^2 - p'^2 + pp'')}{p'^2}\right)$$

$$= Z_i - \frac{p}{p'} \left(1 + \frac{pp''}{2p'^2}\right) \qquad (2.4)$$

Method (2.4) is the Chevbyshev third order method and consider method of Jarrat (1968) for more details.

Next, by setting $\Phi(Z_i) = \frac{p}{p'}\left(1+\frac{pp''}{2p'^2}\right)$, and using as before the procedure described above we write:

$$\hat{Z} = Z_i - \Phi(Z_i)\left(1+\frac{1}{3}\,(g'(Z))\right) \qquad (2.5)$$

We differentiate g (Z) as follows:

$$g'(Z) = \frac{d}{dz}\left\{Z - \left(\frac{p}{p'} + \frac{p''}{2p'}\left(\frac{p}{p'}\right)^2\right)\right\} =$$

$$1 - \left\{\frac{p'^2 - pp''}{p'^2} + \frac{p''}{2p'}\frac{d}{dz}\left(\frac{p}{p'}\right)^2 + \left(\frac{p}{p'}\right)^2\frac{d}{dz}\frac{p''}{2p'}\right\}$$

$$= 1 - \left\{\frac{p'^2-pp''}{p'^2} + \frac{pp''}{P'^2}\left(\frac{p'^2-pp''}{p'^2}\right) + \frac{1}{2}\,\frac{p^2}{p'^2}\,\frac{p'p'''-p''^2}{(p')^2}\right\}$$

After some calculations we shall obtain:

$$1 - \frac{2p'^4 - 4p'^2\,pp'' + 3p^2p''^2 - p^2p''p'''}{2p'^4} \qquad (2.6)$$

Where, the term $-4p'^2\,pp''$ is introduced into the numerator part of (2.6) to compensate for the term $3p^2p'^2 - p^2p'p'''$.

However, the choice of this extral term $-4p'2\,pp''$ is arbitary which is also found to be optimal in some sense.
In view of (2.6), method (2.5) takes the form:

$$\hat{Z}_i = Z_i - \Phi(Z_i)\left(1+\frac{1}{3}\left(1-\frac{2p'^4-4p'^2pp''+3p^2p''^2-p^2p'p'''}{2p'4}\right)\right) \qquad (2.7)$$

Simplifying we have:

$$\hat{Z}_i = Z_i - \Phi(Z_i)\left(1+\frac{1}{3}\left(\frac{4p'^2-3p^2p''^2+p^2p'p'''}{2p'^4}\right)\right) \qquad (2.8)$$

Following carefully some selected ideas in Hansen and Partick (1977), it is easy to see that:

$$\frac{pp''-p'^2}{p^2} = -\sum_i\left(\frac{1}{Z-Z_i}\right)^2$$

By Cauchy - Schwartz inequality, we have that:

$$\left(\sum_i \frac{1}{Z-Z_i}\right)^2 \le n\sum_i\left(\frac{1}{Z-Z_i}\right)^2 \qquad (2.9)$$

It follows that:

$$p'^2 - p\,p'' = p^2\sum\left(\frac{1}{Z-Zi}\right)^2 \ge \frac{p^2}{n}\left(\sum\frac{1}{Z-Zi}\right)^2$$

Deduce then immediately that:

$$(n-1)\,p'^2 - n\,p\,p'' \ge 0 \qquad (2.10)$$

In view of (2.10), we shall obtain two new methods out of method (2.8) which is precisely our aim.

We shall categorize our procedures of derivation under two cases:

Case I: $p\,p'' \le \frac{n-1}{n}\,p'^2$ (2.11)

We now rewrite (2.8) by subsituting the right hand side of (2.11) for pp'' into the resulting expressing for $g'(Z)$:

$$\hat{Z}_i = Z_i - \Phi(Z_i)\left(1+\frac{1}{3}\left(\frac{pp''(4p'^2-3pp'')+p^2p'p''}{2p'^4}\right)\right)$$

$$= Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\,\frac{\left(\frac{n-1}{n}\right)p'^2\left(p'^2-3\left(\frac{n-1}{4n}\right)p'^2\right)+p^2p'p'''}{2p'^4}\right\}$$

$$= Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\left(\frac{\frac{(n-1)}{n}\,p'^4\left(1-3\frac{(n-1)}{4n}\right)+p^2p'p'''}{2p'^4}\right)\right\} \qquad (2.12)$$

We simplify the term as follows:

$$\frac{(n-1)}{n}p'^4\left(1-3\frac{(n-1)}{4n}\right) = \frac{n-1}{4n}p'^4\left(\frac{4n-3n+3}{4n}\right) = \frac{n-1}{n}p'^4\left(\frac{n+3}{4n}\right)$$

$$= \frac{(n-1)(n+3)\,p'^4}{4n^2} \qquad (2.13)$$

Substitute (2.13) into (2.12), we have:

$$\hat{Z}_i = Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\left(\frac{(n-1)(n+3)\,p'^4}{8n^2\,p'4}+\frac{p^2p'p'''}{2P'^4}\right)\right\}$$

$$= Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\left(\frac{(n-1)(n+3)}{8n^2}+\frac{p^2p'p'''}{2P'^4}\right)\right\}$$

$$\hat{Z}_1 = Z_i - \Phi(Z_i)\left\{\frac{p^2p'p'''}{6p'^4}+\frac{(n-1)(n+3)}{24n^2}+1\right\} \qquad (2.14)$$

Case II: $p'^2 \ge \frac{n}{n-1}\,pp''$ (2.15)

As before, we have from:

$$\hat{Z}_1 = Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\,\frac{pp''\left(p'^2\ \frac{3}{4}\ pp''\right)+p^2p'p'''}{2p'^4}\right\} \qquad (2.16)$$

Substitute the right hand side of (2.15) for p'^2 in method (2.16), we have:

$$\hat{Z}_1 = Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\,\frac{\left(\frac{n-1}{n}\right)\left(p'^2-\frac{3\,(n-1)}{4\ \ n}p'^2\right)+p^2p'p'''}{2p'^4}\right\} \qquad (2.17)$$

Simplifying (2.17), we have:

$$\hat{Z}_1 = Z_i - \Phi(Z_i)\left\{1+\frac{1}{3}\left(\frac{\frac{(n-1)}{n}p'^2\left(\frac{4n-3n+3}{4n}\right)+p^2p'p'''}{2p'^4}\right)\right\}$$

Table 1

No. of Iteration	Proposed method 2.14	Proposed method 2.18	Proposed method 2.19	Halley's method	Chevbyshev's method
0	-4.500000000	-4.5000000000	-4.500000000	-4.500000000	-4.500000000
1	-4.036653964	-4.0569632550	-4.010442833	-4.033284200	-4.056976693
2	-3.998106190	-4.0031990970	-4.047858808	-4.000019194	-4.000212678
3	-4.000101004	-3.9999994955	-4.001058584	-4.000012796	-4.000000000
4	-3.999994613	-4.0000000000	-4.000000008	-3.999997013	-
5	-4.000000285		-3.9999999869	-3.999996018	-
6	-3.999999670	-	-	-3.9999994692	-
7	-	-	-	-4.000000002	-
8	-	-	-	-4.0000000000	-

$$= Z_i - \Phi(Z_i)\left(1 + \frac{1}{3}\left(\frac{\left(\frac{(n-1)(n+3)}{4n^2}\right)p'^2 + p^2 p' p'''}{2p'^4}\right)\right)$$

$$= Z_i - \Phi(Z_i)\left(1 + \frac{(n-1)(n+3)\,p'^2 + 12n^2\,p^2\,p'\,p'''}{12n^2 \text{ x } 2p'^4}\right)$$

$$= Z_i - \Phi(Z_i)\left(\frac{p^2\,p'\,p'''}{2p'^4} + \frac{(n-1)(n+3)\,p'^2}{24n^2\,p'^4} + 1\right) \qquad (2.18)$$

Now, if we ignore the extract term $-4p'^2\,pp''$ added to method (2.8) we shall obtain directly an iterative method of the form:

$$\hat{Z}_i = Z_i - \Phi(Z_i)\left(1 + \frac{p^2\,p'\,p''' - 3p'^2\,p''^2}{6p'^4}\right) \qquad (2.19)$$

Methods (2.14), (2.18) and (2.19) are of fourth order of convergence, in the sense of Hansen and Patrick (1977) as well as Traub (1964) and Petkovic and Herceg (1992). In the next section, we shall discuss our methods numerically in comparison with Halley's method, Davies and Dawson (1975), (Hansen and Patrick (1977), Chevbyshev's method, Petkovic and Herceg (1992).

Results and Discussion

The iterative methods discussed earlier can be used for any polynomial of degree $n \geq 3$. Our scalar test problem is a polynomial of degree 5 given by:

$$p(Z) = Z^5 - 6Z^4 - 20Z^3 + 120Z^2 + 64Z - 384 = 0$$

The initial inclusion zeros is $Z^{(0)} = -4.5$

We used the results from our methods (2.14) and (2.18) to compare with results from Halleys method and Chevbyshev's method. The Halley's and Chevbyshev's methods are given by:

$$\hat{Z}_1 = Z_i - \frac{p}{p' - \frac{pp''}{2p'}}$$

(Halley's method)

$$\hat{Z}_1 = Z_i - \frac{p}{p'}\left(1 + \frac{pp''}{2p'^2}\right)$$

(Chevbyshev's method)

We present our results in ordinary real floating arithmetic.

All results are presented below in Table 1.

It can be seen from the Table 1 that Halley's third order method performs worst than any of the four methods. It can also be seen that our method (2.18) has high and extremely fast convergency properties as Chevbyshev's third order method. It is also noticed that our method (2.14) is converging but at a rate less than both of our methods, (2.18), (2.19) and Chevbyshev's method.

The actual zero of the polynomial problem is - 4.

One striking thing about our methods (2.14) and (2.18) is that, convergence to the desired zeros is not affected by the degree of polynomial.

All our tested problems are polynomial with real simple zeros. The methods can be adapted for polynomial with real multiple zeros but this has not been studied in details in this paper.

As, in Patrick and Hansen (1992) iterative method. it is known that Cauchy - Schwartz inequality can be used to prove that:

$$p'^2 - \left(\frac{1}{n-1} + 1\right)pp'' > 0$$

This was a major plank in which our methods were obtained. Since convergence is monotonic for all real zeros satisfying the methods described above, the method with fastest convergence is one with the largest step $Z^{(k+1)} - Z^{(k)}$. As observed in our calculations, it is seen that the factors $\frac{p^2 p' p'''}{2p'^4}$ and $\frac{p^2 p' p'''}{6p'^4}$ decay to zero number at a very rapid rate. It is also hoped that if multiple precision arithmetic is used in the implementation of the methods, the rate of convergency may require fewer steps in our calculations.

Local existence. We set forth to prove that our methods conform with fixed point theorem.

Theorem. Each of our methods (2.14), (2.18) and (2.19) converges monotonically towards the desired zero.

Proof. Since each has a Lipschitz constant less than 1 it is self verifying that each of the methods does not possess normal structure. It follows that the sequence $\{Z_i^{(k)}\} \to \xi$ where, all $Z_i^{(k)}$ and ξ all lie in the interval disk of desired zero. It holds good that in the limit as $k \to \infty$ $p(Z_i^{(k)}) \to 0$. This shows that the distance topology is $\|Z_i^{(k)} - Z_i^{(k+1)}\| \to 0$. In addition, the triangular inequality holds for the set $\{Z_i^{(k)}\}$. Therefore, we conclude that our methods are feasible and endorsed fixed point which converges to a limit point.

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