

COMMON FIXED POINT THEOREMS IN 2-METRIC SPACE

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(Received 25 August 1994; accepted 20 June 1995)

The object of this paper is to prove a fixed point theorem of Singh and Meade type (1977) in 2-metric space.

Key words: Fixed point theorems, Housdroffmetric, 2-Metric space.

Introduction

Gahler (1963) has introduced the notion of a 2-metric space as follows:

A 2-metric space is a non empty set X together with a real valued function d on $X \times X \times X$ satisfies the following properties:

- (i) For two distinct points x, y in X , there exists a point z in X such that $d(x, y, z) \neq 0$.
- (ii) $d(x, y, z) = 0$ if at least two of x, y, z are equal.
- (iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$
- (iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all x, y, z, u in X .

Definition 1.1: sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be Cauchy sequence if $\lim_{m, n \rightarrow \infty} d(x_m, x_n, z) = 0$ for all z in X .

Definition 1.2. A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Lemma 1.1. (Cho et al 1988) For every $i, j, k \in \mathbb{N}$, $d(x_i, x_j, x_k) = 0$ where $\{x_n\}$ is the sequence in X defined in the proof of the theorem.

Results and Discussion

Throughout this paper (X, d) stands for a complete 2-metric space. Further, ϕ is the set of functions $\phi : (\mathbb{R}^+)^5 \rightarrow \mathbb{R}^+$, which are upper semi-continuous from the right and non-decreasing in each coordinate variable such that $\phi(t, t, t, at, bt) < t$ for each $t > 0$ and $a \geq 0, b \geq 0$ with $a + b \leq 2$.

In this note we extend result of Singh and Meade (1977) for 2-metric space. We prove the following:

Theorem 1. Let (X, d) be a complete 2-metric space and let S and T be self mapping of X . Suppose there exists a $\phi \in \Phi$ such that for all $x, y, a \in X$, $d(Sx, Ty, a) \leq \phi[d(x, y, a), d(x, Sx, a), d(y, Ty, a), d(x, Ty, a), d(y, Sx, a)] \dots (2.1)$ holds, then there

exists a $z \in X$ such that z is a unique common fixed point of S and T .

Proof. Let $x \in X$ be any point. Then define a sequence of iterates $\{x_n\}$. First choose a point x_1 in $X_1 = Sx$. Then choose a point x_2 in $X_2 = Tx_1$ and so on. In general choose a point x_{2n-1} in $X_{2n-1} = Sx_{2n-2}$ and a point x_{2n} in $X_{2n} = Tx_{2n-1}$ for $n = 1, 2, 3, \dots$. Put $V_n = d(X_n, X_{n+1}, a)$. We shall prove that $\{V_n\}$ is a decreasing sequence.

If $V_{2n+1} > V_{2n}$ then by (2.1) we have

$$\begin{aligned} V_{2n+1} &= d(X_{2n+1}, X_{2n+2}, a) = d(Sx_{2n}, Tx_{2n+1}, a) \\ &\leq \phi [d(x_{2n}, x_{2n+1}, a), d(x_{2n}, Sx_{2n}, a), d(x_{2n+1}, \\ &Tx_{2n+1}, a), d(x_{2n}, Tx_{2n+1}, a), d(x_{2n+1}, Sx_{2n}, a)] \\ &\leq \phi (V_{2n}, V_{2n}, V_{2n+1}, V_{2n} + V_{2n+1}, 0) \\ &< V_{2n+1} \text{ as } V_{2n+1} > 2V_{2n}. \end{aligned}$$

giving a contradiction.

Hence $V_{2n+1} < V_{2n}$. Similarly we can show that $V_{2n+2} < V_{2n+1}$. Thus $\{V_n\}$ is a decreasing sequence.

Now since

$$\begin{aligned} V_1 &= d(X_1, X_2, a) = d(Sx, Tx_1, a) \\ &\leq (V_0, V_0, V_0, 2V_0, 0). \end{aligned}$$

it follows by induction that

$$\begin{aligned} V_n &\leq \psi^n (V_0), \text{ where} \\ \psi(t) &= \max \{ \phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t) \} \end{aligned}$$

Then by a lemma of Matkowsk (1977) we have

$$\lim_{n \rightarrow \infty} V_n = 0 \dots \dots \dots (2.2)$$

We now show that $\{x_n\}$ is a Cauchy sequence. For this, it suffices to prove that $\{x_{2n}\}$ is a Cauchy sequence.

Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each even integer $2k$, there are even integers $2m(k) > 2n(k) > 2k$ such that

$$d(x_{2m(k)}, x_{2n(k)}, a) > \epsilon \dots\dots\dots(2.3)$$

By the well ordering principle, for each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.3). That is to say that

$$d(x_{2n(k)}, x_{2m(k)-2}, a) \leq \epsilon \text{ and (2.3) holds} \dots\dots(2.4)$$

Now

$$\epsilon < d(x_{2n(k)}, x_{2m(k)}, a) \leq d(x_{2n(k)}, x_{2m(k)-2}, a) + V_{2m(k)-2} + V_{2n(k)-1}$$

(using lemma 1.1)

Then by (2.3) and (2.4) it follows that

$$\lim_{k \rightarrow \infty} d(x_{2n(k)}, x_{2m(k)}, a) = \epsilon \dots\dots\dots(2.5)$$

Also by the triangle inequality and using lemma 1.1 we have

$$|d(x_{2n(k)}, x_{2m(k)-1}, a) - d(x_{2n(k)}, x_{2m(k)}, a)| \leq V_{2m(k)-1}$$

and

$$|d(x_{2n(k)+1}, x_{2m(k)-1}, a) - d(x_{2n(k)}, x_{2m(k)}, a)| \leq V_{2m(k)-1} + V_{2n(k)}$$

Using (2.5) we get

$$d(x_{2n(k)}, x_{2m(k)-1}, a) \rightarrow \epsilon$$

and

$$d(x_{2n(k)+1}, x_{2m(k)-1}, a) \rightarrow \epsilon$$

But by the hypothesis (2.1) one gets

$$d(x_{2n(k)}, x_{2m(k)}, a) \leq d(x_{2n(k)}, x_{2n(k)+1}, a) + d(x_{2n(k)+1}, x_{2m(k)}, a) + d(x_{2n(k)+1}, x_{2m(k)}, a) + d(x_{2n(k)}, x_{2m(k)}, a)$$

The last term will be zero by lemma 1.1

$$= d(x_{2n(k)}, x_{2n(k)+1}, a) + d(x_{2n(k)+1}, x_{2m(k)}, a) \leq V_{2n(k)} + d(Sx_{2n(k)}, Tx_{2m(k)-1}, a),$$

$$\leq V_{2n(k)} + \phi[d(x_{2n(k)}, x_{2m(k)-1}, a) d(x_{2n(k)}, Sx_{2n(k)}, a) d(x_{2m(k)-1}, Tx_{2m(k)-1}, a), d(x_{2n(k)}, Tx_{2m(k)-1}, a), d(x_{2m(k)-1}, Sx_{2n(k)}, a)] \leq V_{2n(k)} + \phi[d(x_{2n(k)}, x_{2m(k)-1}, a), V_{2n(k)}, V_{2n(k)-1}] d(x_{2n(k)}, x_{2m(k)}, a), d(x_{2m(k)-1}, x_{2n(k)+1}, a)]$$

As ϕ is upper semi continuous from the right, we obtain

$$\leq \phi(\epsilon, 0, 0, \epsilon, \epsilon) < \epsilon, \text{ when } k \rightarrow \infty,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. By the completeness of X , $\{x_n\}$ converges to a point $z \in X$.

Now

$$d(z, Sz, a) \leq d(z, x_{2n}, a) + d(z, Sz, x_{2n}) + d(Sz, Tx_{2n-1}, a) \leq d(z, x_{2n}, a) + d(z, Sz, x_{2n}) + \phi[d(z, x_{2n-1}, a), d(z, Sz, a), d(x_{2n-1}, Tx_{2n-1}, a), d(z, Tx_{2n-1}, a), d(x_{2n-1}, Sz, a)] \leq d(z, Sz, x_{2n}) + d(z, x_{2n}, a) + \phi[d(z, x_{2n-1}, a), d(z, Sz, a), V_{2n-1}, d(z, x_{2n-1}, a) + d(x_{2n-1}, Tx_{2n-1}, a) + d(z, Tx_{2n-1}, x_{2n-1}) d(x_{2n-1}, Sz, a)]$$

Letting n tending to ∞ , we obtain

$$d(z, Sz, a) \leq \phi(0, d(z, Sz, a), 0, 0, d(z, Sz, a))$$

By the property of ϕ , we get $d(z, Sz, a) = 0$. Similarly, it can be shown that $d(z, Tz, a) = 0$. Thus $z \in Sz \cap Tz$. This completes the proof.

Remark 1. Taking to be continuous function for metric space we get a slightly revised version of the result of Husain and Sehgal (1975) as a corollary.

References

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