Pak J Sci Ind Res 42 1999 (5) 236-241

MIXED MONOTONICITY THEOREMS FOR A SYSTEM OF ABSTRACT MEASURE DELAY INTEGRO-DIFFERENTIAL EQUATIONS

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(Received 28 September 1993; accepted 27 April 1995)

In this paper the existence of extremal solutions of a system of nonlinear abstract measure delay integro-differential equations is established using the fixed point theorem of Tarski. Two basic integro-differential inequalities are obtained which are further applied to prove the boundedness and uniqueness of the solution of related abstract measure delay integro-differential equations.

Key words: Monotonicity theorems, Integro differential equations, Fixed point theorem.

Introduction

The main feature of monotonicity theorems is to establish the existence of maximal and minimal solutions of related problem under certain monotonicity conditions on the functions involved in it. The monotonicity theorems for abstract measure differential equations were established by (Birkhoff 1967) Joshi and Deo (1980) and Joshi and Kasralikar (1982). Also in (Dhage 1989) the present author obtained similar results for certain abstract measure integro-differential equations. In 1990, present author (Dhage 1990) considered a system of abstract measure delay integro-differential equations (in short delay AMIDE) which was a generalization of ordinary delay integro-differential equations and studied the basic problems such as existence, uniqueness, extention and stability using the fixed point techniques. In the present study we exploit the same delay AMIDE for other aspects of the solutions namely, maximal and minimal solutions and the boundedness of the solutions. As mentioned earlier (Dhage 1990) the delay AMIDE is more general and includes several measure differential equations as special cases.

The paper is presented in sections describing notations and preliminaries needed in the sequel, the statement of the problem and the existence theorem for extremal solutions, the integro-differential inequalities for the related delay AMIDE and finally the applications of the delay integro-differential inequalities are discussed.

Notations and preliminaries. Let R denote the real line, R^n the Euclidean space with respect to the norm |.| defined by

$$|x| = |x_1| + ... + |x_n|_s \text{ .... } (2.1)$$

for $x = (x_1,..., x_n) \in R^n$.

Let X be a Banach space with a norm denoted by || . ||. For any two points $x, y \in X$, the sequent $\overline{xy}$ is defined by

$$\overline{xy} = \{ z \in x \mid z = x + r(y-x), 0 \le r < 1\} \text{....} (2.2)$$

Suppose that x_0 and y_0 are two fixed points of X, and z a variable point of X such that $\overline{x_0z}$ and $\overline{y_0z}$ are non-empty and $\overline{x_0z} \subset \overline{y_0z}$. For $x_1, x_2 \in \overline{y_0z}$, we write $x_1 < x_2$ (or $x_2 \ge x_1$) if $\overline{y_0x_1} \in \overline{y_0x_2}$. For any point $x \in \overline{y_0z}$, define the sets S_x and $\bar{S}_x$ as follows:

$$S_x = \{rx \mid -\infty < r < 1\}, \quad \bar{S}_x = \{ rx \mid -\infty < r \le 1\} \text{.....} (2.3)$$

The distance $\| x_0 - y_0 \|$ between x_0 and y_0 is denoted by w, i.e. $\| x_0 - y_0 \| = w$. For each $x \in \overline{x_0z}$ there exists a unique vector x' such that x' < x and || x-x' || = w. This vector is denoted by x_w.

A vector measure P defined on a 6-algebra M means an ordered n-tuples $(P_1,...,P_n)$ of n real measures (finite signed measures). The norm $\| P \|_n$ on P is defined by

$$\| P \|_n = \| P_1 \| + ... + \| P_n \| \text{ .... } (2.4)$$

where $\| P_i \|$ denotes the usual norm of the real measure P_i. Let the space of all vector measures defined on M by denoted by ca (X, M). It can be shown that ca(X, M) is a Banach space with respect to the norm defined by (2.4) Dunford and Schwartz (1958). If μ is a positive measure on M, and P ∈ ca (X, M), P is absolutely continuous w.r.t. μ, if μ(E)=0 implies P(E)=0 (the zero vector in R^n). In this case P << μ. For P ∈ ca (X, M), a positive measure $|P|_n$ is defined by

$$| P |_n (E) = \sum_{i=1}^{n} | P_i | (E) \text{ .... } (2.6)$$

where $| P_i |$ denotes the total variation measure of the real measure P_i. It is known that $| P(E) |_s \le | P |_n (E)$, E ∈ M.

Let M_0 denote the smallest σ-algebra on Sx_0 containing $\{x_0\}$ and the sets $\overline{S}_x$, $x \in \overline{y_0x_0}$. For any $z > x_0$, let M_z denote the smallest σ-algebra defined on S_z containing M_0 and the sets $\overline{S}_x$, $x \in \overline{x_0z}$. For a given positive number H, the sets B_H and C_H are defined by

$$B_H = \{ u \in R^n \mid |u|_s < H \} \quad \text{.......(2.7)}$$

and

$$C_H = \{ q \in ca(Sx_0, M_0) \mid \| q \|_n + C < H, C > 0\} \quad \text{.........(2.8)}$$

Delay amide and extremal solutions. For $P << \mu$, we consider the delay AMIDE, involving the delay w,

$$\frac{dP}{d\mu} = f(x, P(\overline{S}_x), P(\overline{S}x_w)$$

$$+ \int_{F(Sx)} k(F, y) P(y, P(\overline{S}_y), P(\overline{S}y_u) d\mu \quad \text{.......(3.1)}$$

$F \subset \overline{x_0z}$, $F \in M_z$,

satisfying the initial conditions

$$P(E) = q(E), E \in M_0 \quad \text{.......(3.2)}$$

where $q \in C_M$ is a known vector measure, $dP/d\mu$ is the Radon-Nikodym derivative of P with respect to the positive real measure μ. f(x, y, z) and g(x, y, z) are the R^n-valued functions defined on $S_z \times B_H \times B_H$ such that for each $P \in ca(S_z, M_z)$, f(x, $P(\overline{S}_x)$, $P(\overline{S}x_w)$ and g(x, $P(\overline{S}_x)$, $P(\overline{S}x_w)$ are μ-integrable, and k(F, x) is a real n x n matrix defined on $M_z \times S_z$. The detaile of the delay AMIDE (3.1) - (3.2) and its special forms are given in the literature (Dhage 1990).

Definition 3.1. Given an initial measure $q \in C_H$, a vector measure $\rho \in ca(S_z, M_z)$ (for some $z > x_0$) is said to be a solution of delay AMIDE (3.1)-(3.2), if

i) $P(E) = q(E)$, $E \in M_0$, (ii) $P << \mu$ on $\overline{x_0z}$, (iii) $P(E) \in B_H$, $E \in M_z$, (iv) P satisfies (3.1) a.e. [μ] on $\overline{x_0z}$.

A solution P of (3.1)-(3.2), existing on $\overline{x_0z}$ will be denoted by $P(\overline{S}x_0, q)$.

Remark 3.1. The conditions (ii) and (iv) are together equivalent to the condition

$$P(E) = \int f(x, P(\overline{S}_x), P(\overline{S}x_w)\, d\mu$$

$$+ \int_E (\int_{F(Sx)} k(F, y)\, g(y, P(\overline{S}y, P(\overline{S}y_w)\, d\mu)\, d\mu \quad \text{........(3.3)}$$

$E \in \overline{x_0z}$, $E \in M_z$.

Now an order relation α in R^n is introduced as follows.

Let $x = (x_1,...,x_n)$ and $y = (y_1,...,y_n)$ be two elements in R^n. Suppose k is a fixed positive integer such that $k < n$. Then by $x \,\alpha\, y$, we mean $x_i \le y_i$ if $i = 1,2,...,k$ and $x_i \ge y_i$ if $i = k+1,...,n$. The definitions of the extremal solutions of AMIDE are given as follows (3.1)-(3.2).

Definition 3.2. A solution $P_M = P_M(\overline{S}x_0, q)$ of the AMIDE (3.1)-(3.2), existing on $\overline{x_0z}$ $(z > x_0)$ is said to be maximal, if for any other solution $P = P(Sx_0, q)$ of (3.1)-(3.2), $P(E) \,\alpha\, P_M(E)$, $E \in M_z$. Similarly a minimal solution P_m of the AMIDE (3.1)-(3.2) may be defined.

Remark 3.2. By the nature of the order relation α, the maximal and minimal solutions defined above are respectively k-max, (n-k)-mini and (n-k)-max, k-mini solutions of (3.1)-(3.2) in the sense of definition considered by Deo and Murdheshwer (1972).

Lemma 3.1. Let $S_0 = ca(X, M)$ be the set of all vector measures defined on the σ-algebra M. Then (S_0, α) is a complete lattice.

Proof. Let $S_0 = S_1 \times S_2 \times ... \times S_n = ca(X, M)$. Then $P \in S_0$ implies $P = (P_1,...,P_n) \in S_1 \times S_2 \times ... \times S_n$ and $P_i \in S_i$, where each S_i, $i = 1,2,...,n$ is a Banach space of real measure with a norm $|P_i|$ defined as the total variation measure of P_i. It can be proved by the arguments similar to those used in Deo and Murdheshwer (1972) that each S_i is a complete lattice w.r.t. the order relation ≤ if $i = 1,2,...,k$ and w.r.t. the order relation ≥ if $i = k+1,...,n$. Since the product of complete lattices is a complete lattice, (S_0, α) is a complete lattice.

We need the following key theorem due to Tarski (1955) in the sequel:

Theorem A. Let (L, α) be a complete lattice and let T be an isotone increasing mappings on L into itself. Then the set $F = \{ u \in L \mid Tu = u \}$ is non-empty and (F, α) is a complete lattice.

Remark 3.3. A mapping T on a lattice L with order relation is said to be isotone increasing if x, y ∈ L, $x \,\alpha\, y$ implies $Tx \,\alpha\, Ty$.

The folllowing assumptions are made:

(A_1) $\mu(\{x_0\}) = 0$

(A_2) f(x, y, z) and g(x, y, z) are nondecreasing functions in y and z with respect to the order relation α, for each $x \in S_z$.

(A_3) There exist the non-negative μ-integrable real functions $W_1(x)$ and $W_2(x)$ defined on S_z, $z > x_0$ such that $|f(x, y, z)|_s \le W_1(x)$ and $|g(x, y, z)|_s \le W_2(x)$ uniformly for $y, z \in B_H$.

(A_4) The matrix k(F, x) is non-negative i.e. each element of n x n matrix K(F, x) is a non-negative real number and

$\sup_{F \in M} \int_E |k(F, x)|\, d\mu \le K_0$ for all $x \in S_z$, where |.| denotes a suitable matrix norm.

Theorem 3.1. Suppose that the assumptions (A_1)-(A_4) are satisfied and $q \in C_H$. Then there exist maximal and minimal solutions of the AMIDE (3.1)-(3.2) on $\overline{x_0 z}$ for some $z > x_0$.

Proof. Let $\{r_n\}$ be a decreasing sequence of real numbers such that $r_n \to 1$ as $n \to \infty$, and

$$Sr_1x_0 \supset Sr_2x_0 \supset \ldots \supset Sx_0,$$

Then it follows that:

$$\lim_{n\to\infty} (\{Sr_nx_0 - \overline{S}x_0\}) = 0 \qquad (3.4)$$

This shows that there exist a number r and a point $x_1 = rx_0$ such that $Sx_0 \subset Sx_1$ and

$$\int_{\overline{x_0x_1}} W_1(x)\, d\mu + K_0 \int_{\overline{x_0x_1}} W_2(x)\, d\mu < H - \| q \|_n \qquad (3.5)$$

This is possible by hypothesis (A_2) and positiveness of μ. A subset S is defined in the Banach space $ca(Sx_1, Mx_1)$, by

$$S = \{P \in ca(Sx_1, Mx_1) \mid P(E) = q(E), E \in M_0,$$

$$\text{and } \| P \|_n \leq K\} \qquad (3.6)$$

where $K = \| q \|_n + \int_{\overline{x_0x_1}} W_1(x)\, d\mu + K_0 \int_{\overline{x_0x_1}} W_2(x)\, d\mu$

It follows from (3.5) and (3.6) that $\| P \|_n < H$, for $P \in S$. Now if the operator T on S is defined by

$$T(P)(E) = q(E),\ E \in M_0 \qquad (3.7)$$

$$T(P)(E) = \int_E f(x, P(\overline{S}x), P(\overline{S}x_w)\, d\mu$$
$$+ \int_E (\int_{E(Sx)} k(F, y)\, g(y, P(\overline{S}y), P(\overline{S}y_w)\, d\mu)\, d\mu \qquad (3.8)$$

$F \in \overline{x_0x_1}$, $E \in Mx_1$.

Then as shown in Theorem 1 of Dhage (1990), the operator T maps complete lattice, it is complete w.r.t. the order relation α. Let $P_1, P_2 \in S$ and $P_1 \alpha P_2$, then by (A_2), we get

$$TP_1(E) = \int_E f(x, P(\overline{S}x), P(\overline{S}x_w)\, d\mu$$
$$+ \int_E (\int_{F(Sx)} k(P, y)\, q(y, P_1(\overline{S}y), P_1(\overline{S}y_w))\, d\mu)\, d\mu$$

$$\alpha \int_E f(x, P_2(\overline{S}x), P_2(\overline{S}x_w))\, d\mu$$
$$+ \int_E (\int_{F(Sx)} k(F, y)\, q(y, P_2(\overline{S}y), P_2(\overline{S}y_w))\, d\mu)\, d\mu$$

$$= TP_2(E),\ E \in \overline{x_0x_1},\ E \in Mx_1$$

This shows that the operator T is an isotone increasing on S. An application of theorem A yields that the fixed point set of the operator T is non-empty and complete lattice. Consequently the solution set of the AMIDE (3.1)-(3.2) is non-empty and complete lattice. This further implies that the delay AMIDE (3.1)-(3.2) has maximal and minimal solutions on $\overline{x_0x_1}$, $x_1 > x_0$. The proof is complete.

Integro-differential Inequalities. The basic inequalities concerning the solution of the integro-differential inequalities are established as follows. They also serve as the bound for the solution of the related integro-differential equations and are useful for proving the uniqueness of the solutions of the delay AMIDE (3.1)-(3.2).

Theorem 4.1. Let the assumptions of theorem 3.1 be satisfied. Suppose that a $\phi \in S$, where S is defined as in theorem 3.1, satisfies

$$\phi(E)\, \alpha\, q(E),\ E \in M_0 \qquad (4.1)$$

$$\frac{d\phi}{du}\, \alpha\, f(x, \phi(\overline{S}x), \phi(\overline{S}x_w))$$
$$+ \int_{E(Sx)} k(F, y)\, g(y, \phi(\overline{S}y), \phi(\overline{S}y_w)\, d\mu \qquad (4.2)$$

$F \in \overline{x_0x_1}$, $F \in Mx_1$.

Then

$$\phi(E)\, \alpha\, P_M(E),\ E \in Mx_1 \qquad (4.3)$$

where $P_M = P_M(\overline{S}x_0, q)$ is the maximal solution of (3.1)-(3.2) existing on $\overline{x_0x_1}$, $x_1 > x_0$

Proof. Let $P = \sup S$. Clearly the element P exists since S is a complete lattice. Consider the lattice interval $[\phi, P]$, which is obviously a complete lattice. Define the operator T on S as in the proof of theorem 3.1. Then T is isotone increasing and maps S into itself. To show that T maps $[\phi, P]$ into itself, it is engough to prove that if $P \in S$, and $\phi\, \alpha\, P$ then $\phi\, \alpha\, Tp$. Let $E \in Mx_1$, $E \in \overline{x_0x_1}$, then using (4.1), (4.2) and the assumption (A_2), the following is obtained:

$$\phi(E)\, \alpha \int_E f(x, \phi(\overline{S}x), \phi(\overline{S}x_w)\, d\mu$$
$$+ \int_E (\int_{F(Sx)} k(F, y)\, g(y, \phi(\overline{S}y), \phi(\overline{S}y_w))\, du)\, d\mu$$
$$\alpha \int_E f(x, p(\overline{S}x), p(\overline{S}x_w)\, d\mu$$
$$+ \int_L (\int_{F(Sx)} k(F, y)\, g(y, p(\overline{S}y), p(\overline{S}y_w))\, du)\, d\mu$$

$$= Tp(E)$$

Thus it is proved that T maps $[\phi, P]$ into itself. An application of theorem A gives that the maximal solution P_M of (3.1)-(3.2) lies in $[\phi, P]$. This implies that $\phi(E)\, \alpha\, P_M(E)$, $E \in Mx_1$. This completes the proof.

Theorem 4.2. Let the assumptions of theorem 3.1 be satisfied. Suppose that the function $\phi \in S$, where S is defined as in the proof of theorem 3.1, satisfies

$$q(E)\ \alpha\ \psi(E),\ E \in M_0 \qquad (4.4)$$

$$f(x, \psi(\bar{S}x), \psi(\bar{S}x_w)) + \int_{F(Sx)} k(F, y)\, g(y, p\,\psi(\bar{S}y), \psi(\bar{S}y_w))\, d\mu\ \alpha\ d\psi/du \qquad (4.5)$$

$E \in \overline{x_0x_1}$, $E \in Mx_1$.

Then

$$P_m(E)\ \alpha\ \psi(E),\ E \in Mx_1 \qquad (4.6)$$

where P_m is the minimal solution of the AMIDE (3.1)-(3.2) existing on $\overline{x_0x_1}$, $x_1 > x_0$.

The proof of theorem 4.2 is similar to theorem 4.1 and the details are omitted.

In the following section, the applications of the integro-differential inequalities are given to prove the boundedness and uniqueness of the solution of the AMIDE (3.1)-(3.2) which may be viewed as the comparison theorems for the AMIDE (3.1)-(3.2).

Applications. Consider the delay AMIDE, involving the delay w.

$$dr/du = u(x, r(\bar{S}x), r(\bar{S}x_w)) + \int_{F(Sx)} h(F, y)\, v(x, r(Sy), r(Sy_w))\, d\mu \qquad (5.1)$$

$E \in \overline{x_0x_1}$, $E \in Mx_1$,

satisfying the initial condition

$$r(E) = q_0(E),\ E \in M_0 \qquad (5.2)$$

where u(x, y, z), v(x, y, z) are non-negative μ-integrable functions defined on $Sx_1 \times R^+ \times R^+$ (R^+ being the set of al positive real numbers), h(F, y) is a positive real function defined on $Mx_1 \times Sx_1$, r, μ are finite positive measures on Mx_1 and $q_0 \in C_H$ is an initial positive measure defined on M_0.

A monotonicity theorem similar to theorem 3.1 for the equation (5.1)-(5.2) can be proved on similar lines. It is merely stated without proof.

Theorem 5.1. Let all the assumptions of 4.1 hold, with f, g and k being replaced by u, v and h respectively. Let $r_M = r_M(\bar{S}x_0, q_0)$ be the maximal solution of (5.1)-(5.2) existing on $\overline{x_0x_1}$. Suppose $\phi \in S$, where S is defined as in theorem 4.1, satisfies

$$|\phi|_n(E) \le q_0(E),\ E \in M_0 \qquad (5.3)$$

$$d|\phi|_n/d\mu \le u(x, |\phi|_n(\bar{S}x), |\phi|_n(\bar{S}x_u)) + \int_{F(Sx)} h(F, y)\, v(y, |\phi|_n(\bar{S}y), |\phi|_n(\bar{S}y_w))\, d\mu \qquad (5.4)$$

$E \in \overline{x_0x_1}$, $E \in Mx_1$

Then

$$|\phi|_n(E) \le r_M(E),\ E \in Mx_1 \qquad (5.6)$$

Theorem 5.2. Let all the assumptions of theorem 5.1 hold. Assume further that the functions f,g and k occuring in (3.1)-(3.2) satisfy

$$|f(x, y, z)|_s \le u(x, |y|_s, |z|_s)$$
$$|g(x, y, z)|_s \le v(x, |y|_s, |z|_s) \qquad (5.7)$$

for all $(x, y, z) \in Sx_1 \times P_H \times B_H$, and

$$|K(F, x)| \le h(F, x) \qquad (5.8)$$

for $(F, x) \in Mx_1 \times Sx_1$. If $P(\bar{S}x_0, q)$ is any solution of (3.1)-(3.2) satisfying

$$|q(E)|_s \le q_0(E),\ E \in M_0 \qquad (5.9)$$

then

$$|P(E)|_s \le r_M(E),\ E \in Mx_1 \qquad (5.10)$$

where r_M is the maximal solution of (5.1)-(5.2).

Proof. If $P(\bar{S}x_0, q)$ is a solution of (3.1)-(3.2), then

$$P(E) = q(E),\ E \in M_0,$$

and $P(E) = \int_E f(x, P(\bar{S}x), P(\bar{S}x_w))\, du + \int_E \left(\int_{F(Sx)} k(F, y)\, g(y, P(\bar{S}x), P(\bar{S}x_w))\, d\mu\right) d\mu$

$E \subset x_0x_1$, $E \in Mx_1$

This by virtue of (5.7)-(5.8), definition of $|P|_n$ and the increasing character of μ and v, implies that

$$|P(E)|_s \le u(x, |P|_n(\bar{S}x), |P|_n(\bar{S}x_w))\, d\mu + \int_E \left(\int_{F(Sx)} h(P, y)\, v(y, |P|_n(Sy), |P|_n(\bar{S}y_w))\, d\mu\right) d\mu \qquad (5.11)$$

Further the inequalities (5.3)-(5.4) imply that

$$|P|_n(E) \le \int_E u(x, |P|_n(\bar{S}x), |P|_n(\bar{S}x_w))\, d\mu + \int_E \left(\int_{E(Sx)} h(f, y)\, v(y, |P|_n(\bar{S}y), |P|_n(\bar{S}y_w))\, d\mu\right) d\mu$$

Since $|P(E)|_s \le |P|_n(E)$, $E \in Mx_1$, an application of theorem 4.1 yields that

$$|P(E)|_s \le r_M(E),\ E \in Mx_1$$

This completes the proof.

Theorem 5.3. Let all the assumptions of theorem 5.1 hold. Suppose that f and g satisfy

$$|f(x, y_1, z_1) - f(x, y_2, z_2)|_s \le u(x, |y_1 - y_2|_s, |z_1 - z_2|s$$

$$|g(x, y_1, z_1) - g(x, y_2, z_2)|_s \le v(x, |y_1 - y_2|_s, |z_1 - z_2|s \quad \text{........(5.12)}$$

for $(x, y_1, z_1), (x, y_2, z_2) \in Sx_1 \times B_H \times B_H$, and the condition (5.7) holds. Further suppose that the zero measure is the only solution of (5.1) with q_0 identically zero measure. Then the equation (3.1) - (3.2) has at the most one solution.

Proof. Let $P_1 = P_1(\bar{S}x_0, q)$ and $P_2 = P_2(\bar{S}x_0, q)$ be any two solutions of the AMIDE (3.1)-(3.1) on x_0x_1, $x_1 > x_0$. Then

$$P_1(E) = P_2(E) = q(E), \text{ if } E \in M_0, \text{ and}$$

$$P_1(E) - P_2(E) = \int_E [\, f(x, P_1(\bar{S}x), P_1(\bar{S}x_w) - f(x, P_2(\bar{S}x), P_2(\bar{S}x_w))]\, d\mu$$

$$+ \int_E (\int_{F(Sx)} k(f, y)\, [\, g(x, P_1(\bar{S}x), P_1(\bar{S}x_w) - f(x, P_2(\bar{S}x), P_2(\bar{S}x_w))]\, d\mu$$

$$E \subset \overline{x_0x_1},\ E \in Mx_1$$

This, by virtue of (5.7) and (5.12), the definition of positive measure and increasing nature of u, v imply that

$$|P_1(E) - P_2(E)|_s \le \int_E u(x, |P_1 - P_2|_n(\bar{S}x), |P_1 - P_2|(\bar{S}x_w))\, d\mu$$

$$+ \int_E (\int_{E(Sx)} h(F, y)\, v(y, |P_1 - P_2|_n(\bar{S}y), |P_1 - P_2|(\bar{S}y_w))\, d\mu)\, d\mu$$

Now an application of theorem 5.1 yields that

$$|P_1(E) - P_2(E)|_s \le 0,\ E \in Mx_1$$

Hence $P_1(E) = P_2(E)$, $E \in Mx_1$

The proof is complete.

Discussion. The equation (3.1)-(3.1) includes several measure differential equations discussed earlier by different authors as particular cases. If $g = 0$ and $n = 1$, then the abstract measure delay differential equation is obtained as considered by Joshi and Deo (1980).

$$\left.\begin{aligned} &\frac{dP}{d\mu} = f(x, P(\bar{S}x), P(\bar{S}x_w)) \\ &\text{and} \\ &P(E) = q(E),\ E \in M_0 \end{aligned}\right\} \quad \text{..............................(6.1)}$$

Again when $g = u$ and $f(x, P(\bar{S}x), P(\bar{S}x_w)) = f(x, P(\bar{S}x_w))$ in (3.1) the delay differential equation is obtained as considered by Joshi and Kasralikar (1982).

$$\left.\begin{aligned} &\frac{dP}{d\mu} = f(x, P(\bar{S}x_w)) \\ &P(E) = q(E),\ E \in M_0 \end{aligned}\right\} \quad \text{......................................(6.2)}$$

Similarly if $w = 0$, F is independent of Sx and k(F, x) is a real-valued function in (3.1), then it gets reduced to the abstract measure integro-differential equation studied by the present author (Dhage 1989).

$$\left.\begin{aligned} &\frac{dP}{d\mu} = f(x, P(\bar{S}x)) + \int_F k(F, x)\, g(x, P(\bar{S}x)))d\mu \\ &P(E) = q(E),\ E \in M_0. \end{aligned}\right\} \quad \text{...........(6.3)}$$

The AMIDE (6.3) further includes the AMDE discussed by Sharma (1975).

$$\left.\begin{aligned} &\frac{dP}{d\mu} = f(x, P(\bar{S}x_w)) \\ &P(\bar{S}x_0) = \lambda,\ \lambda \in R \end{aligned}\right\} \quad \text{...(6.4)}$$

Thus the AMIDE (3.1)-(3.2) is more general and hence the results of this paper include the results of Dhage (1989), Joshi and Deo (1980), Joshi and Kasralikar (1982) and Sharma (1975) as special cases.

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