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SOME NEW EXACT SOLUTIONS OF EQUATIONS OF MOTION OF AN INVISCID COMPRESSIBLE FLUID VIA ONE PARAMETER GROUP

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Some new exact solutions of equations governing the flow of an inviscid compressible fluid are determined using one parameter group for an arbitrary state equation.

Key words: Equation of motion, Inviscid fluid, One parameter.

Introduction

The present work describes some new exact solutions of equations governing the flow of an inviscid compressible fluid for an arbitrary state equation. The steady plane flow of an inviscid compressible fluid, in the absence of heat conduction, is governed by the system of five non-linear partial differential equations [1].

$$(\rho u)_x + (\rho v)_y = 0$$

$$\rho u u_x + \rho v u_x = -P_x$$

$$\rho u v_x + \rho v v_x = -P_y \quad \text{.......(1)}$$

$$u s_x + v s_y = 0$$

$$P = P(\rho, s)$$

where u, v are the velocity components, P the pressure, ρ the density, and s the specific entropy of the fluid. On introducing

$$\rho u = \psi_y$$

$$\rho v = -\psi_x \quad \text{.......(2)}$$

the system (1) is replaced by the following system

$$P_\xi = 8\,(R_\eta \psi_\xi \psi_\eta + R\psi_\xi \psi_{\eta\eta} - R_\xi \psi^2_\eta - R\psi_\eta \psi_{\xi\eta}) \quad \text{.......(3)}$$

$$P_\eta = 8\,(R_\xi \psi_\xi \psi_\eta + R\psi_\eta \psi_{\xi\xi} - R_\eta \psi^2_\xi - R\psi_\xi \psi_{\xi\eta})$$

$$\psi_\xi s_\eta = \psi_\eta s_\xi$$

$$P = P(\rho, s)$$

of four equations in four unknown functions ψ, P, R, s, as functions of ξ(= x + y) and η(= x - y). In above, ψ is the streamfunction and the function R is given by

$$R = 1/\rho \quad \text{.......(4)}$$

Once a solution of system of equations (3) is determined, the density ρ is determined from equation (4).

One-parameter group and transformation of flow equations. In this section, system (3) of non-linear partial differential equations is transformed [2] into a new system of ordinary differential euqations using one-parameter group of transformations. General group theory that is used to determine some exact solutions of system (3) is given here briefly [2-6].

If Γ_1 is a group consisting of a set of transformations defined by

$$\bar{\xi} = a^n \xi,\ \bar{\eta} = a^m \eta,\ \bar{\psi} = a^k \psi$$

$$\bar{P} = a^t P,\ \bar{R} = a^j R,\ \bar{s} = a^q s$$

with parameter $a \neq 0$, then the invariants of group Γ_1 for system (3) are

$$R = \eta^{\alpha_1} H(\theta),\ \psi = \eta^{\alpha_2} Q(\theta),\ P = \eta^{\alpha_3} T(\theta),\ \theta = \xi/\eta$$

$$s = \eta^{\alpha_4} V(\theta) \quad \text{.......(5)}$$

provided

$$\alpha_3 = 2\alpha_2 + \alpha_1 - 2 \quad \text{.......(6)}$$

In equation (5)

$$\alpha_1 = j/n,\ \alpha_2 = k/n,\ \alpha_3 = t/n,\ \alpha_4 = q/n$$

The invariants (5) of Γ_1 transform system (3) into the following system of ordinary differential equations in the four unknown function H, Q, T, V of θ are:

$$T' = 8[H\{(1-\alpha_2)\theta Q'^2 + \alpha_2 \theta Q Q'' + \alpha_1 Q'(\alpha_2 Q - \theta Q')\} + H'\{\alpha_2 Q(\theta Q' - \alpha_2 Q)\}] \quad \text{.......(7.1)}$$

$$\alpha_3 T - \theta T' = 8\,[H\{\alpha_2 Q Q'' - (\alpha_2 - 1)Q'^2 - \alpha_1 Q'^2\} + \alpha_2 Q Q' H'] \quad \text{.......(7.2)}$$

$$\alpha_4 VQ' = \alpha_2 QV' \quad \text{.......(7.3)}$$

Once a solution of the system (7) of ordinary differential equations is determined, the streamfunction ψ, the density ρ, the entropy s and pressure P are determined employing equations (4-5).

Solutions of the flow equations. The solutions of the system (7) consisting of three ordinry differential equations have been determined here. In order to determine the solutions of system (7) for an arbitary state equation, the system (7) becomes underdetermined. However, system can be determined by assuming the function H to be constant or by finding a differential euqation for the function H and then applying on it the particular methods for determining the solutions of ordinary differential equations to obtain forms for the function Q.

The differential equation for H is obtained by eliminating the function T from equations (7.1-7.2)

When $\alpha_3 = 0$, the equations (7.1-7.2), on eliminating T, give

$$(D + B\theta)H' + (A\theta + C)H = 0 \quad \text{.......(8)}$$

This is the required differential equation for H for $\alpha_3 = 0$ when $\alpha_3 \neq 0$, equation (7.2), on using equation (7.1), gives

$$\alpha_3 T = 8\,[(A\theta + C)H + (D + \theta B)H'] \quad \text{.......(9)}$$

Equation (9) and (7.1) imply that

$$(D+\theta B)H'' + (\theta A+\theta B'+C+D'+(1-\alpha_3)B]H' + [\theta A+C'+(1-\alpha_3)A]H = 0 \quad \text{.......(10)}$$

where in equation's (8-10),

$$A(\theta) = (1-\alpha_1-\alpha_2)\theta Q'^2 + \alpha_2\theta QQ'' + \alpha_1\alpha_2 QQ'$$

$$B(\theta) = -\alpha_2 Q + (\alpha_2 Q - \theta Q')$$

$$C(\theta) = \alpha_2 QQ'' + (1-\alpha_1-\alpha_2)Q'^2$$

$$D(\theta) = \alpha_2 QQ'$$

Equation (10) is the required differential equation for the function H for $\alpha_3 \neq 0$

Integration of equations (7.1) and (7.3) yield

$$T = 8\int [AH - BH']\,d\theta + C_1 \quad \text{.......(11.1)}$$

$$V = C_2 Q^{\alpha_4/\alpha_2},\ \alpha_2 \neq 0 \quad \text{.......(11.2)}$$

where C_1 and C_2 are arbitrary constants.

The equations (11.1) and (11.2) give the general expressions for the functions T and V for the flow equations. To determine the solutions of equations (8) and (10), equations (8) and (10) are treated separately as two cases, case-I and case-II, respectively.

Case I: when $\alpha_3 = 0$

The differential equation (8) is a linear differential equation in H and the function H can easily be determined by integrating it once. The integration of (8) for

$D + \theta B \neq 0$, yields

$$H = A_1 \exp \int \frac{(A\theta + C)}{(D + \theta B)}\,d\theta \quad \text{.......(12)}$$

where A_1 is an arbitrary constant. In equation (12) the function $Q(\theta)$ on which the functions A, B, C, D, depend is such that

$$Q' \neq 0 \quad \text{.......(13)}$$

and

$$Q \propto \theta^{\alpha_2} \quad \text{.......(14)}$$

The reason that the function $Q(\theta)$ can not have the form defined in equations (13-14) is that these forms of $Q(\theta)$ give u=v. Hence $P_x = P_y$ and P can not be a function of θ only unless a trivial constant.

Since the function $Q(\theta)$ in equation (12) can assume a large number of forms, except the forms defined by equations (13-14), a large number of solutions of the flow equations for the case $\alpha_3=0$ can be constructed.

However, in some cases, forms for the function $Q(\theta)$ can be determined by assuming appropriate forms for the function $H(\theta)$ in equation (8), e.g. equation (9), on using

$H = \theta^m$, gives

$$(1-\alpha_1-\alpha_2)\,\theta(1+\theta^2)Q'^2 + \alpha_2(\alpha_1+m)\theta^2 QQ' + m\alpha_2 QQ' + \alpha_2\theta(1+\theta^2)QQ'' - m\alpha_2^2\theta Q^2 = 0 \quad \text{.......}(q_1)$$

This differential equation has a solution when

$1 - \alpha_1 - \alpha_2 = 0$ which is determined as follows:

Recalling that for $\lambda_3 = 0$, α_1 and α_2 satisfying the equation

$$2\alpha_2 + \alpha_1 = 2$$

This equation and the equation $1 - \alpha_1 - \alpha_2 = 0$ give

$\alpha_1 = 0, \qquad \alpha_2 = 1$

The differential equation (q_1), using α_1=0, α_2=1, gives

$$\theta(1+\theta^2)Q'' + m(1+\theta^2)Q' - m\theta Q = 0$$

Substituting $Q = \theta^2 X(\theta)$ in the above equation,

$$\theta^3(1+\theta^2)X'' + (4+m)(\theta^2+\theta^4)X' + [2(1+m)\theta + (m+2)\theta^3] X = 0$$

which can be rewritten as

$$\theta^3(1+\theta^2)X'' + [1+m)\theta^2 + (m+3)\theta^4]X' + (3\theta^2+\theta^4)X' + [2(1+m)\theta + (m+2)\theta^3] X = 0$$

The right hand side of the above differential equation becomes an exact differential when m = 2. Taking m = 2,

$$[\theta^3(1+\theta^2)X' + \theta^2(\theta^2+3)X]' = 0$$

Which on integrating once provides

$$\theta^3(1+\theta^2)X' + \theta^2(\theta^2+3)X = D_1$$

wherein D_1 is an arbitrary constant. This is a linear differential equation in X whose solution is given by

$$X = D_1\left[\frac{(1+\theta^2)}{2\theta^3}\tan^{-1}\theta + \frac{1}{2\theta^2}\right] + \frac{D_2(1+\theta^2)}{\theta^3}$$

wherein D_2 is an arbitrary constant. Using this expression of X in $Q=\theta^2 X(\theta)$, we get

$$Q(\theta) = \frac{D_1}{2} + \frac{D_1}{2\theta}(1+\theta^2)\tan^{-1}(\theta) + \frac{D_2(1+\theta^2)}{\theta}$$

When the function H is constant, the equation (8) gives

$$(1-\alpha_1-\alpha_2)(1+\theta^2)Q'^2 + \alpha_2(1+\theta^2)QQ'' \alpha_1\alpha_2\theta QQ' = 0 ......(q_2)$$

Equation (q_2) is a non-linear differential equation in Q. To determine the solutions of the equation (q_2), multiply equation (q_2) by the factor $(1+\theta^2)^\lambda$, which gives

$$(1-\alpha_1-\alpha_2)(1+\theta^2)^{\lambda+1}Q'^2 + \alpha_2 Q[(1+\theta^2)^{\lambda+1}Q'' + \alpha_1\theta(1+\theta^2)^\lambda Q' = 0$$

which can be rewritten as

$$\frac{Q'}{Q} = \left[\frac{2-\alpha_1}{\alpha_1}\right]\frac{[(1+\theta^2)^{\alpha_1/2}Q']'}{(1+\theta^2)^{\alpha_1/2}Q'}, \qquad 2 - \alpha_1 \neq 0 \text{ provided}$$

$$\lambda = (\alpha_1/2) - 1$$

This equation, on integration, provides

$$\ln Q = \frac{2-\alpha_1}{\alpha_1}\ln[(1+\theta^2)^{\alpha_1/2}Q'] + D_3$$

where D_3 is an arbitrary constant. The above equation, by taking $D_3 = \ln D_4$, gives

$$Q = D_4[(1+\theta^2)^{\alpha_1/2}Q']^{(2-\alpha_1)/\alpha_1}$$

where D_4 (>0) is a constant. When $\alpha_1 \neq 1$, the solution is

$$Q^{2(\alpha_1-1)/(\alpha_1-2)} = 2\frac{2(\alpha_1-1)}{(\alpha_1-2)}D_4^{\alpha_1/(\alpha_1-2)}\int(1+\theta^2)^{-\alpha_1/2}d\theta + \frac{2(\alpha_1-1)}{(\alpha_1-2)}D_5$$

where D_5 is an arbitrary constant and $\alpha_1 \neq .2$. The indefinite integral in the above expression for Q can easily be calculated using table of indefinite integrals [7] for given α_1. Since α_1 can assume infinite many values, infinite number of expressions for Q can be constructed.

When α_1=1, the solution is

$$Q = D_6\left[[\theta + (1+\theta^2)^{1/2}\right]^{D_7/2}\exp\left[D_7[\theta(1+\theta^2)^{1/2}]^{/2}\right]$$

where D_6 and D_7 are arbitrary constants.
Adding and subtracting the term $\alpha_1(1+\theta^2)Q'^2$ in equation (q_2) give

$$\alpha_2(1+\theta^2)Q'^2 + \alpha_1(1+\theta^2)QQ'' + (1-\alpha_1-2\alpha_2)(1+\theta^2)Q'^2 + \alpha_1\alpha_2\theta QQ' = 0$$

Which, on rearranging the terms, gives

$$\frac{[QQ']'}{QQ'} = -\frac{\alpha_1}{2}\frac{[(1+\theta^2)Q]'}{(1+\theta^2)Q}$$

provided $\alpha_1\alpha_2 = 2(1-\alpha_1-2\alpha_2)$

This, after twice integration, gives

$$Q^{(3+\alpha_1)/2} = \frac{2}{(3+\alpha_1)} D_8 \int (1+\alpha^2)^{-\alpha_1/2}\, d\theta + \frac{2}{(3+\alpha_1)} D_9$$

where D_8 and D_9 are arbitrary constants and $\alpha_1 = \frac{1 \pm \sqrt{5}}{2}$

The case $D + \theta B = 0$, for equation (8) is discarded since it leads to the trivial solution.

Case II. When $\alpha_1 \neq 0$

The function H in equation (10) is determined by employing the particular methods for determining the solution of second order ordinary differential equations as follows:
The equation (10) is an exact differential equation provided

$$\alpha_3(B' - A) = 0 \qquad (15)$$

Since α_3 can not be zero, therefore

$$B' = A \qquad (16)$$

Equation (16), utilizing the definition of the functions A and B, gives

$$(1+\alpha_3)(-\theta Q' + \alpha_2 Q)Q' = 0$$

This holds for all θ provided

$$\alpha_3 = -1 \text{ or } Q' = 0 \text{ or } -\theta Q' + \alpha_2 Q = 0$$

The last two cases are discarded due to the same reasoning as attributed to the equations (13-14). When $\alpha_3 = -1$ the function $Q(\theta)$ is arbitrary and α_1, α_2 satisfy

$$2\alpha_2 + \alpha_1 = 1 \qquad (17)$$

The first integral of equation (10) is

$$(D + \theta B)H' + (\theta A + C + B)H = b_1 \qquad (18)$$

where b_1 is an arbitrary constant. When $D + \theta B \neq 0$, the general solution of (18) is

$$H = e^{-F_1(\theta)} \left[\int \frac{b_1}{D + \theta B} e^{+F_1(\theta)} d\theta + b_2 \right] \qquad (19)$$

where $F_1(\theta) = \int \frac{\theta A + C + B}{D + \theta B} d\theta$

and b_2 is an arbitrary constant. In equation (19), the function $Q(\theta)$ is arbitrary function and therefore, a large number of solutions of equation (18) can be constructed. For example, taking

$$Q(\theta) = a\,\theta^m,\ m \neq \alpha_2,$$

the equation (19) gives

$$H = \frac{b_1}{\alpha_2 a^2} \theta^{-A_1}[Z(\theta)]^{-A_3} \int \theta^{A_1 - 2m + 1} [Z(\theta)]^{A_3 - 1} d\theta$$

$$+ b_2 (\alpha_2 a^2)^{-A_3} \theta^{-A_1} [Z(\theta)]^{-A_3} \qquad (20)$$

wherein $Z(\theta) = m + (m - \alpha_2)\theta^2$ and $a \neq 0$, and m are arbitrary constants.

The indefinite integrals in equation (20) can easily be evaluated using table of indefinite integrals [7] for given m and α_2. When m is negative, $\alpha_2 \leq m$ in equation (20). In equation (20).

$$A_1 = \frac{-m + (2m-1)\alpha_2}{\alpha_2}$$

$$A_2 = a^2\,[2m^2 - 2\alpha_2^2 - m^2\alpha_1 + m\alpha_1\alpha_2]$$

$$A_3 = \frac{A_2}{2a^2\,\alpha_2\,(m - \alpha_2)}$$

when $D + \theta B = 0$,

$$Q = b_3\,(1+\theta^2)^{\alpha_2/2} \qquad (21)$$

where b_3 is an arbitrary constant. The streamfunction ψ for Q given by equation (21) is

$$\psi = b_3\, 2^{\alpha_2/2}\,(x^2 + y^2)^{\alpha_2/2},$$

and ψ = constant represent a family of concentric circles defined by

$$x^2 + y^2 = \frac{1}{2}\left[\frac{\text{constant}}{b_3}\right]^{2/\alpha_2}$$

The equation (18), utilizing equation (21) gives

$$H = \frac{b_1}{\alpha_2^2 b_3^2 [\{1+(2\alpha_2+2\alpha_1-3)\theta^2\}(1+\theta^2)^{\alpha_2-2}+\theta^2(2+\alpha_1\theta)(1+\theta^2)^{\alpha_2-1}-(1+\theta)^{\alpha_2}} \quad .....(22)$$

when H is constant, the equation (10) gives

$$[(1-\alpha_1-\alpha_2+\alpha_1\alpha_2)+(1-\alpha_3)(1-\alpha_1-\alpha_2)]\theta Q'^2+$$
$$(2-2\alpha_1-\alpha_2)(1+\theta^2)Q'Q''+\alpha_2(\alpha_1-\alpha_3+2)\,\theta\, QQ''+$$
$$\alpha_2(1+\theta^2)QQ'''+\alpha_1\alpha_2(1-\alpha_3)QQ'=0 \quad ....(23)$$

Using $\alpha_3 = 2\alpha_2+\alpha_1-2$ in the above equation,

$$[(1-\alpha_1-\alpha_2+\alpha_1\alpha_2)+(3-\alpha_1-2\alpha_2)(1-\alpha_1-\alpha_2)]\theta Q'^2+$$
$$(2-2\alpha_1-\alpha_2)(1+\theta^2)Q'Q''+\alpha_2(4-2\alpha_2)\,\theta\, QQ''+\alpha_2(1+\theta^2)QQ'''$$
$$+\alpha_1\alpha_2(3-\alpha_1-2\alpha_2)QQ'=0 \quad ....(24)$$

A solution of this non-linear differential equation for

$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = 1$, is

$$Q = 2(1+\theta^2) \quad ....(25)$$

The streamfunction ψ for Q defined by (25) is

$$\psi = 4(x^2+y^2)(x-y)^{-1}$$

For ψ = constant = e_1, $x \neq y$, the streamline are circles defined by

$$\left[x+\frac{e_1}{8}\right]^2+\left[y-\frac{e_1}{8}\right]^2=\frac{e_1^2}{32}$$

Note that when x=y the streamfunction is singular i.e. it becomes infinite at x=y. Physically it means that there are either sources or sinks or both situated on the line y=x. Most of the solutions that have been presented for the cases α_3=0 and $\alpha_3 \neq 0$, contain the arbitrary function $Q(\theta)$ and this arbitrariness of $Q(\theta)$ enables to construct a large number of solutions. Now some forms for $Q(\theta)$ are given which represent physically possible flow situations. For example taking $Q(\theta)$ equal to $\beta_1+\beta_2\theta^\lambda+\beta_3(1+\theta^2)$ and $(\beta_4-\beta_5\theta)[\beta_6+\beta_7/(1+\theta^2)$ and appropriately choosing the constants $\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7,\lambda$, give (i) flows with stramline a fmily of straight lines inclined to the x-axis, (ii) flows with streamlines a family of ellipses, (iii) flow with streamlines a family of hyperbolae, (iv) flow with streamlines a family of concentric circles, (v) Hiemenz flow, and (vi)flow due to a doublet situated at the origin

Conclusions

Employing one parameter group, some new exact solutions of a system of partial differential equations governing the motion of an inviscid compressible fluid are determined for an arbitrary equation of state. When the expression $D(\theta)+\theta B(\theta)$ is non-zero, the solutions for α_3=0 and $\alpha_3 \neq 0$ contain the arbitrary function $Q(\theta)$which enables to construct a large number of solutions to the flow equations. The work presented herein increases the number of known exact solutions to the flow equations and provides new types of singular solutions admitted by the flow equation which are not determinable through other methods.

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