Physical Sciences Section

Pakistan J. Sci Ind. Res., Vol. 31, No. 7, July 1988

GROUP CALIBRATION OF MASSES

Part I. Masses of Denomination 10, 5, 2, 2′, 1′, Kg

M. A. Chaudri and M. Arif

National Physical and Standard Laboratory, Islamabad

(Received April 7, 1988; revised July 12, 1988)

Some basic aspects of calibration of the working standard masses have been explained. The system of group weighing, formulation of least square equations and calculations of the mass values of the unknown masses in terms of the certified reference standard 1 kg mass have been illustrated. It has also been shown how the different types of variance, standard deviation and uncertainties can be determined from the systematic uncertainty of the reference standard and the random errors of measurement.

Keywords : International system, Calibration, Least square solution.

INTRODUCTION

Mass in one of the seven base units in the International System (SI) of Weights and Measures. The unit of mass, the international prototype kilogram, is a unique artifact in the custody of International Bureau of Weights and Measures (BIPM) at Sevres in France. This is a cylindrical piece made of 90% platinum and 10% iridium with height equal to diameter. The signatories of the convention of the metre have a right to acquire a copy of International kilogram duly calibrated and numbered by the BIPM. Pakistan has yet to order and acquire a copy of the prototype as its national kilogram. The National Physical and Standards Laboratory (NPSL), Islamabad is however in possession of two internationally calibrated sets of stainless steel masses with denominations ranging from 1 mg to 20 kg. One set is preserved as the primary standard and the other is being used as the reference standard. This is our intention to highlight, through this manuscript, how the calibration of masses should be done at the highest level of accuracy. The results are then disseminated to various organizations interested in the higher accuracy of mass measurement. This system results in a hierarchy of mass measurement from the international kilogram down to the commercial weight.

For the present, only one scheme is discussed to obtain accurate values for a set of masses from 1 kg to 10 kg in terms of a 1 kg reference standard. It is also shown how the uncertainties may be calculated.

EXPERIMENTAL

The problem of determining the values of a set of masses starting with a known mass of 1 kg is dealt with here. Masses are compared in combinations so that the masses in each pan of the balance are nominally equal, and the unknown masses are calculated in terms of those that are known. In our case, the value of 1 kg alongwith its uncertainty is known through the certificate issued by the National Standard Laboratory of Belgium as 1.0000014 kg ± 0.0000003 kg (± 300 μ g).

The procedure consists of forming the best combination that will give the least uncertainty with the minimum number of weighings. There is no unique solution to this problem but a number of satisfactory solutions are available in the literature on this topic [2, 3, 4, 5, 6].

The mass values of a set consisting of nominal values 10, 5, 2, 2′, 1′, kg have been calculated in terms of the certified 1 kg reference standard by weighing the following combinations by the method described elsewhere by Chaudri *et al* [1]. These are called observational equations and the number of equations must exceed the number of unknown parameters to affect a least-squares solution.

[5]	[2]	[2′]	[1] (Reference)	[1′]		
1	1	1	1		=	1-+a_1
1	1	1		1	=	10+a_2
1	−1	−1	−1		=	a_3
1	−1	−1		−1	=	a_4
	1	−1	1	−1	=	a_5
	1	−1	−1	1	=	a_6
	1	−1			=	a_7
	1	−1			=	a_8
	1		−1	−1	=	a_9
	1		−1	−1	=	a_{10}
		1	−1	−1	=	a_{11}
		1	−1	−1	=	a_{12}

1	−1	=	a_{13}
1	−1	=	a_{14}

The values of $a_1, a_2 \ldots\ldots\ldots a_{14}$ are observed mass differences, buoyancy corrections where necessary may be applied.

The normal equations for the system are :

$$4\ [5] = 2\ [10] + a_1+a_2+a_3+a_4 \quad \ldots\ldots\ldots\ldots \quad (1)$$

$$10\ [2] = 2\ [10] + a_1+a_2-a_3-a_4+a_5-a_6+a_7+a_8+a_9a_{10} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$$10\ [2'] = 2\ [10] + a_1+a_2-a_3-a_4-a_5-a_6-a_7-a_8-a_9a_{11}+a_{12} \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$10\ [1] = [10] + a_1-a_3+a_5-a_6-a_9-a_{10}+a_{11}-a_{12}+a_{13}+a_{14} \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$10\ [1'] = [10] + a_2-a_4-a_5+a_6-a_9-a_{10}-a_{11}-a_{12}a_{13}-a_{14} \quad \ldots\ldots\ldots\ldots \quad (5)$$

Calculations. The value of the heaviest mass [10] can be calculated from equation (4) in terms of the certified value of the reference standard 1 kg mass, as

$$[10] = [10]\ 1 - a_1+a_3-a_5+a_6+a_9+a_{10}+a_{11}+a_{12}-a_{13}-a_{14} \quad \ldots\ldots\ldots\ldots \quad (6)$$

Having calculated the value of [10], we can calculate the values of the masses of denomination [5], [2], [2′], [1′] kg.

A practical example to illustrate the whole procedure is given below. First, all the 14 observational equations are solved and the values of the differences $a_1, a_2 \ldots\ldots a_{14}$ are determined by the method described in detail previously [1]. In this way, we obtain the following matrix :

	[5]	[2]	[2′]	[1] (Reference)	[1′]		
1.	1	1	1	1	0	=	[10] – 0.0445
2.	1	1	1	0	1	=	[10] + 0.0775
3.	1	−1	−1	−1	0	=	– 0.0390
4.	1	−1	−1	0	−1	=	+ 0.0235
5.	0	1	−1	1	−1	=	– 0.0485
6.	0	1	−1	−1	1	=	– 0.0420
7.	0	1	−1	0	0	=	+ 0.0015
8.	0	1	−1	0	0	=	+ 0.0035
9.	0	1	0	−1	−1	=	– 0.0070
10.	0	1	0	−1	−1	=	– 0.0080
11.	0	0	1	−1	−1	=	– 0.0480
12.	0	0	1	−1	−1	=	– 0.0500
13.	0	0	0	1	−1	=	– 0.0035
14.	0	0	0	1	−1	=	+ 0.0040

By substituting the certified value of the 1 kg reference standard in equation 4 above, we get the following values of all the masses in the group from the normal equation 1–6 above :

[10]	= 10	[1] – 0.1015	= 9999.9125
[5]	= 1/4	[2 (10) + 0.0175]	= 4999.9606
[2]	= 1/10	[2 (10) – 0.0520]	= 1999.9773
[2′]	= 1/10	[2 (10) + 0.0360]	= 1999.9861
[1′]	= 1/10	[10] + 0.1730	= 1000.0086

The residual errors and their squares have been calculated below by substituting the values of [10], [5], [2], [2′], [1], [1′] in the 14 observational equations given above in order to calculate the uncertainties of measurement in terms of the uncertainty of the certified 1 kg reference standard mass.

Residuals	Squares
+ 0.0575	0.003295
– 0.0575	0.003295
+ 0.0348	0.001211
– 0.0349	0.001218
+ 0.0325	0.001056
+ 0.0404	0.001632
– 0.0103	0.000106
– 0.0123	0.000151
– 0.0257	0.000660
– 0.0247	0.000610
– 0.0241	0.000581
+ 0.0261	0.000681
– 0.0037	0.000014
– 0.0112	0.000125

Sum of residuals squares = 0.014635, and the variance $\sigma^2 = \frac{\Sigma\,(\text{residuals})^2}{(n-k)}$, where n is the number of observational equations and k is the number of unknown parameters.

$$\sigma^2 = \frac{0.014635}{9}$$

$$= 1.62611 \times 10^{-3}$$

$= 162611 \times 10^{-8}$ and the standard deviation

$\sigma = 403.25 \times 10^{-4}$ is equal to the square root of

$\sigma = 403.25 \times 10^{-4}$ is equal to the square root of the variance

Standard deviation of the 1 kg reference standard can be calculated from the relationship of standard deviation with the uncertainty which is usually taken as uncertainty = 3 x standard deviation. Hence standard deviation σ of 1 kg reference standard $= \frac{\text{uncertainty}}{3}$, or $\sigma = \frac{300\,\mu g}{3}$ and Var [1] = $1 \times 10^{-8}\ g^2$.

The variance of each calibrated mass has been calculated using the following standard equation for such problems [2] :

$$\mathrm{Var}(f) = \left(\frac{\partial f}{\partial a}\right) \mathrm{Var}(a) + \left(\frac{\partial f}{\partial b}\right)^2 \mathrm{Var}(b) + \ldots\ldots +$$

$$2\frac{\partial f}{\partial a}\cdot\frac{\partial f}{\partial b}\,\mathrm{Cov}(a,b) + 2\frac{\partial f}{\partial a}\,\frac{\partial f}{\partial c}\,\mathrm{Cov}(a,c,) + \ldots\ldots$$

Where f is a function of the parameters a,b,c ......, var (a) is the variance of a, and cov (a,b) is the covariance between a and b.

In our treatment, however, the covariance between various parameters is zero as they are independent of each other.

From equations 1 – 6,

since $[10] = 10\,[1] - a_1 + a_3 - a_5 + a_6 + a_9 + a_{10} + a_{11} + a_{12} - a_{13} - a_{14}$,

hence Var [10] = 10^2 Var [1] + 10 σ^2 = 162610×10^{-8}

Var [5] = $(¼)^2$ [(2)^2 Var (10) + 4σ^2] = 447205×10^{-8}

Var [2] = $(1/10)^2$ [(2) Var (10) + 10σ^2] = 81309×10^{-8}

Var [2′] = 81309×10^{-8}

Var [1′] = $(1/10)^2$ [Var (10) + 10σ^2] = 32523×10^{-8}

The calibrated values of the masses in terms of the certified reference standard mass of 1 kg alongwith different types of variance, standard deviation and the uncertainty are listed in Table 1. Total uncertainty has been obtained by simple addition of the random uncertainty and the systematic uncertainty as done by Prowse and Anderson [7].

CONCLUSIONS

The values of the calibrated working standard masses are well within the permissible limits of error as provided in the Pakistan Weights and Measures (International System) rules, 1974 published in the Gazette of Pakistan (Extraordinary) Islamabad May 25, 1974. The scheme of experimentation is very useful as it provides a reliable measure of the random uncertainty alongwith the calibrated values in terms of the reference standard mass.

Similarly, the masses of denomination 500, 200, 200′, 100, Σ100g can be calibrated in terms of the 1 kg certified reference standard mass. Then the 100 g mass can be used as the reference standard to calibrate the 50, 20, 10, Σ10 g masses and the 5,2,2′,1, Σ1 g masses can be calibrated against a reference standard mass of 10 g and the 1 g reference standard mass can be used to calibrate the mg series of masses.

The value of the mass shown against the summation sign Σ equals the lowest mass value in a given group of masses under calibration and will be formulated by a suitable combination of the lower denomination masses in the next decade, for instance Σ100 g = 50 g + 20 g + 20′ g + 10 g.

The scheme will therefore be further utilized to calibrate masses of lower denomination in future.

Table 1. Calibrated mass values and their uncertainty

Nominal mass (g)	Observed mass (g)	Observed variance x10^{-8} (g)^2	Standard deviation (g)	Random uncertainty (g)	*Systematic uncertainty (g)	Total uncertainty ±(g)
10,000	9,999.9125	1626210	0.1275	0.3825	0.0030	0.3855
5,000	4,999.9606	447205	0.0669	0.2007	0.0015	0.2022
2,000	1,999.9773	81309	0.0285	0.0855	0.0006	0.0861
2′,000	1,999.9861	81309	0.0285	0.0855	0.0006	0.0861
1,000	–	–	–	–	0.0003	0.0003 (Certified reference standard of 1 kg, as mentioned in the text).
1′,000	1,000.0086	32523	0.0188	0.0564	0.0003	0.0567

*The uncertainty of the 1 kg reference standard mass is a systematic uncertainty.

Acknowledgement. The authors are thankful to Dr. A. Kalim for his help in the statistical treatment of the data and improvement in the manuscript.

REFERENCES

1. M.A. Chaudri, Shaheen Raja and A.K. Javed, Pakistan J.Sci. Ind. Res., **27**, 275 (1984).
2. D.B. Prowse, Proceedings Asia Pacific Metrology Programme, **2**, 33 (1982).
3. P.E. Pontius and J.M. Cameron, NBS Monograph, 103 (1967).
4. W. Wats on, *A Text Book of Practical Physics* (Longmans, Green and Co. London, 1930), 3rd ed. pp, 63-78.
5. M. Zuker, G. Mihailov and M. Romanowski, Metrologia **16**, 51 (1980).
6. S.V. Gupta, MAPAN, **1**, 35 (1986).
7. D.B. Prowse and A.R. Anderson, Metrologia, **10**, 123 (1974).