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INTERACTION OF MHD SHOCKS

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A plane MHD shock wave of arbitrary strength meets a slender body moving at supper-true-sonic speed in the opposite direction. The interaction between the given shock wave and the weak shock attached to the slender body is studied for aligned fields for axisymmetrical flow and for both aligned and transverse fields in the two-dimensional case. Formal solutions for the linearized flow in the interaction region are obtained by the use of integral transforms.

INTRODUCTION

Suppose a plane shock wave of arbitrary strength meets a slender body which is moving in the opposite direction at supersonic speed. Since the slender body has a weak attached shock wave, the problem ultimately involves an interaction between the two shock waves. This problem, in different contexts, has been considered by many authors, using a variety of approaches. In particular, Arora [1], whose paper may be consulted for additional references, used integral transforms to obtain an analytical solution to a linearized formulation of the stated problem for slender axisymmetrical bodies and two-dimensional thin airfoils.

The purpose of the present paper is to extend Arora's approach to the case of a perfectly conducting fluid. For the axisymmetric case, the problem of an aligned magnetic field is considered, so that the shock transition relations are merely the usual ones from conventional gas dynamics, supplemented by the continuity of the normal component of the magnetic field across the shock.

For the problem of a two-dimensional airfoil, both aligned and transverse magnetic fields are considered. Superposition of these solutions would, of course, yield the behaviour for an arbitrary orientation of the applied magnetic field. Structurally, the solution for the aligned fields case is identical to the solution for the axisymmetric case. With a transverse magnetic field, however, the normal magnetohydrodynamic shock conditions must be utilized. Althoughh the perturbed form of these relations has been given previously [2], a derivation in a form more suitable for the present problem is given in Appendix C.

In Arora's analysis, the ultimate differential equations for the perturbed pressure transforms are second-order ordinary differential equations with constant though literal (the transform parameters) coefficients, and these may be solved exactly, so that the transforms may be inverted. In the present paper, the coupling of magnetic and fluid dynamical effects leads to similar equations, but these are of the fourth-order. Consequently, fourth-order algebraic equations with literal coefficients must be solved in order to obtain the solutions of the ordinary differential equations. Although there are standard techniques for solving such algebraic equations, the resulting expressions for the solutions are quite involved; thus, attention will be focused on special values for two of the parameters which measure the strength of the incident shock and the strength of the applied magnetic field. Complete and exact solutions are given for the pressure perturbation transforms for this special case; the corresponding quartic algebraic equations are solved in Appendices A and B. The final solutions, valid for arbitrary bodies or airfoils consistent with the basic assumptions, are given as formal inversions, since an exact inversion seems unlikely.

The physical formulation of the problem is as follows. A plane shock of arbitrary strength moves with super-true-sonic velocity V in the direction of the positive x-axis of a cylindrical coordinate system into a uniform region (o) of fluid at rest; the uniform velocity in the region (1) behind the shock is U. At time $t = 0$, the shock strikes the apex of an axisymmetric body (insulator) of infinite length moving in the opposite direction with super-true-sonic speed W. The region (2) behind the attached shock is uniform. Thus, when $t \leq 0$, the flow pattern consists of three uniform regions; region (0) in front of and region (1) behind the impinging shock; region (0) in front of and region (2) behind the attached shock.

For $t > 0$, the shocks intersect, and the problem is to determine the flow, within a linearized framework, in region (1) behind the intersecting shocks. A cylindrical coordinate system (r, θ x), fixed relative to the undisturbed flow behind the incident shock will be employed. It is assumed that the diffracted shock is only slightly deflected from its undiffracted position and further meets the body surface normally so that the fluid flow across the shock will remain parallel to the surface. A diagram, illustrating the geometry of the reflected and diffracted waves, of the flow pattern developed in the conventional case is given in Arora[1], Fig.2. Qualitatively, the results will be quite similar in the present case, but, since the analysis does not depend on this diagram, it will not be included here. In region(2), the solution is known from previous work as an isentropic perturbation of an initially uniform flow, while in region (1), the solution must be obtained as a non-isentropic perturbation of an initially uniform flow.

With c the conventional sound speed, the physical quantities defining the problem are $M = V/c_0$, the Mach number of the shock, $M' = W/c_0$, the Mach number of the body and $r = f(\chi)$, the function defining the surface of the body where χ denotes the axial coordinate with origin at the nose of the body. Magnetic parameters will be defined in the body of the text.

Part I. Axisymmetric Flow

The Basic Equations. With respect to an (r, θ x) cylindrical coordinate system, the equations which govern the unsteady axisymmetrical flow of an ideal, inviscid, perfectly conducting compressible fluid, subjected to an applied magnetic field $\vec{B} = (B_1, B_2, B_3)$ may be written as

$$P = \exp\left[\frac{s - s_0}{c_v}\right] \rho^{\gamma} \tag{2.1}$$

$$r\,\rho_t + (\rho r u)_r + r(\rho q)_x = 0 \tag{2.2}$$

$$\rho[u_t + u\,u_r + q\,u_x] + P_r = \frac{B_3}{\mu}(B_{1x} - B_{3r}) - \frac{B_2}{\mu r}(r\,B_2)_r \tag{2.3}$$

$$B_1 (r\,B_2)_r + r\,B_3 B_{2x} = 0 \tag{2.4}$$

$$\rho(q_t + u\,q_r + q\,q_x) + P_x = -\frac{B_2 B_{2x}}{\mu} - \frac{B_1'}{\mu}(B_{1x} - B_{3r}) \tag{2.5}$$

$$(r\,B_1)_r + r\,B_{3x} = 0 \tag{2.6}$$

$$B_{1t} + (q\,B_1 - u\,B_3)_x = 0 \tag{2.7}$$

$$B_{2t} + (q\,B_2)_x + (u\,B_2)_r = 0 \tag{2.8}$$

$$r\,B_{3t} + [r(u\,B_3 - q\,B_1)]_r = 0 \tag{2.9}$$

$$s_t + u\,s_r + q\,s_x = 0 \tag{2.10}$$

where p, ρ, s, s_0, γ, (u, 0, q), c, μ, $b_i^2 = B_i^2/\mu\rho$ (i = 1,2,3), t are, respectively, the pressure, density, specific entropy, at some reference state, ratio of specific heat at constant pressure c_p and at constant volume c_v, velocity vector, local speed of sound, permeability, square of the Alfven speed and time; the subscripts r, x and t denote, partial differentiation in the radial and axial directions and with respect to time. In the next section, these equations will be linearized in the neighbourhood of an initially uniform flow for the special case of aligned magnetic and velocity fields.

Linearized Equations of Motion. A formal linearization of Eqs. (2.1) – (2.10) in the neighbourhood of an initially uniform state with velocity zero (because of the choice of the moving coordinate system) and only the third component of the magnetic induction nonzero leads to the following system of linear equations for the perturbed flow

$$\frac{\rho_t}{\rho} + \overline{q_r} + \frac{\overline{q}}{r} + \overline{u_x} = 0 \tag{3.1}$$

$$\overline{q_t} + \frac{\overline{\rho_r}}{\rho} + b_3^2\left(\frac{\overline{B_{1x}}}{B_3} - \frac{\overline{B_{3r}}}{B_3}\right) \tag{3.2}$$

$$\rho\,\overline{u_t} + \overline{P_x} = 0 \tag{3.3}$$

$$\frac{\overline{B_{1t}}}{B_3} - \overline{q_x} = 0 \tag{3.4}$$

$$\frac{\overline{B_{3t}}}{B_3} + \overline{q_r} + \frac{\overline{q}}{r} = 0 \tag{3.5}$$

$$\frac{\overline{B_{3x}}}{B_3} + \frac{\overline{B_{1r}}}{B_3} + \frac{\overline{B_1}}{rB_3} = 0 \tag{3.6}$$

$$\overline{s_t} = 0, \text{ or } \overline{P_t} - c^2\,\overline{\rho_t} = 0 \tag{3.7}$$

$$\overline{B_{2t}} = 0, \ \overline{B_{2x}} = 0 \tag{3.8}$$

where perturbations are denoted by a bar. Equation (3.8) shows that $B_2 \equiv 0$ since it was zero initially.

Defining $\bar{\rho}/\rho$=A, $\overline{B_3}/B_3$ =B, $\bar{p}/\gamma p$=P, $\bar{q}/c$=Q, $\bar{u}/c$=D and ξ=ct gives the system of equations

$$A_\xi + Q_r + D_x + \frac{Q}{T} = 0 \tag{3.9}$$

$$Q_\xi + P_r = \frac{b_3^2}{c^2}\left(\frac{B_{1x}}{B_3} - B_r\right) \tag{3.10}$$

$$D_\xi + P_x = 0 \tag{3.11}$$

$$\frac{\overline{B_{1\xi}}}{B_3} - Q_x = 0 \tag{3.12}$$

$$B_\xi + Q_r + \frac{Q}{r} = 0 \tag{3.13}$$

$$P_\xi - A_\xi = 0 \tag{3.14}$$

$$B_x = -\left(\frac{\overline{B_{1r}}}{B_3} + \frac{\overline{B_1}}{rB_3}\right) \tag{3.15}$$

which may be combined to yield the following system of coupled linear partial differential equations

$$P_{\xi\xi} = P_{xx} + P_{rr} + \frac{P_r}{r} + m^2\left(B_{xx} + B_{rr} + \frac{B_r}{r}\right) \tag{3.16}$$

$$B_{\xi\xi} = P_{rr} + \frac{P_r}{r} + m^2\left(B_{xx} + B_{rr} + \frac{B_r}{r}\right) \tag{3.17}$$

where m^2 - b_3^2/c^2 In the non-magnetic case, a single wave equation for the pressure perturbation was obtained

The Initial and Boundary Conditions. For the present problem the usual shock transition relations from conventional gas dynamics apply with the addition of the continity of the component of the magnetic field normal to the shock (i.e., B_3) across the shock. With the subscripts 1 and 2 denoting, respectively, the flow behind and in front of the shock, the perturbed form of the shock conditions may be written as

$$A = L_{11}u^* + L_{12}p^* + L_{13}\psi_\xi \tag{4.1}$$

$$P = L_{21}u^* + L_{22}p^* + L_{23}\psi_\xi \tag{4.2}$$

$$D = L_{31}u^* + L_{32}p^* + L_{33}\psi_\xi \tag{4.3}$$

$$Q = L_{41}q^* + L_{42}\psi_r \tag{4.4}$$

together with

$$B = 0 \tag{4.5}$$

where $\overline{P_2}/\gamma P_0$ = P*, $\overline{u_2}/V$ = u*, $\overline{q_2}/V$ = q* and ψ(r,t) denotes the displacement of the shock path from its undisturbed position. With a different notation, the constants are defined in Arora's paper, which may be consulted for details. Alternatively, a derivation of the perturbed flow of the shock relations for a magnetohydrodynamic shock wave propagating into a moving fluid are given in Appendix C; these equations include the usual gas dynamic conditions as the special case obtained in the limite of vanishing magnetic field.

The shock conditions will be applied at the undisturbed shock locus, x = Vt, which in the moving coordinate system is

$$x = a\,\xi \tag{4.6}$$

where a = M c_0/c_1 – n_1, M = V/c_0, n_1 = U/c_1. Elimination of ψ and ψ_ξ from Eqs. (4.1) – (4.4) gives

$$D = \frac{L_{33}}{L_{23}}\left[P + \left(\frac{L_{23}L_{31}}{L_{33}} - L_{21}\right)u^* + \left(\frac{L_{32}L_{23}}{L_{33}} - L_{22}\right)p^*\right]$$
$$\equiv (P - L_{52}u^* - L_{53}p^*)/L_{51} \tag{4.7}$$

$$Q_\xi = L_{41}q_\xi^* + (P_r - L_{21}u_r^* - L_{22}p_r^*)/L_{54}$$
where
$$L_{54} = L_{23}/L_{42} \tag{4.8}$$

In order to write the boundary conditions on the body surface, account must be taken of the solution in the insulator. For the perturbed flow in front of the shock it has been shown that the perturbed magnetic field, whose components are a harmonic vector, vanishes in the insulator [3,4,5]; a simple extension of this reasoning shows that the same situation will be true behind the shock. Continuity shows that all magnetic field pertubrations vanish on the suface of thebody in the fluid behind the shock.

On the surface of the body, the boundary condition for tangency of the flow may be written as

$$rQ = a_1 f(x + a_1\xi) f'(x + a_1\xi) \tag{4.9}$$

where r = f is the cross sectional radius of the body and a_1= M'c_0/c_1 + n_1, M' = W/c_0 Using the second momentum equation, Eq. (3.10), it follows that as r → 0,

$$r P_r + m^2 r B_r \to -a_1^2 F[x + a_1\xi] \tag{4.10}$$

where

$$F(\lambda) = f(\lambda)\, f''(\lambda) + [f'(\lambda)]^2$$

On the body surface along the shock,

$$rQ = a_1 f[(a + a_1)\xi]\, f'[(a + a_1)\xi],$$

So that using Eq. (4.8)

$$r P_r \to L_{54}\, a_1\, (a + a_1)\, F[(a + a_1)\xi] +$$

$$\lim_{r\to 0} \{L_{21}\, r\, u_r^* + L_{22}\, r\, p_r^* - L_{41}\, L_{54}\, r\, q_r^*\} \quad (4.11)$$

along the shock as $r \to 0$ Eq. (4.11) ensures that the flow remains tangential to the body at the base of the shock. Finally, all perturbations are to vanish at infinity, i.e. as $x \to -\infty$, $r \to \infty$, P, B and their derivatives vanish.

For the flow ahead of the shock, the perturbed flow andperturbed magnetic field problems uncouple compeltely [5,6]; thus, the perturbations at the shock may be written as

$$u^* = -k_1 \int_0^{h(\beta r)} \frac{F(\theta)\, d\theta}{[h^2(\theta) - \beta^2 r^2]^{1/2}} \quad (4.12)$$

$$q^* = \frac{k_1}{r} \int_0^{h(\beta r)} \frac{h(\theta) F(\theta)\, d\theta}{[h^2(\theta) - \beta^2 r^2]^{1/2}}$$

$$p^* = -\frac{k_2}{k_1} u^* \quad (4.13)$$

where $k_1 = M'/M$, $k_2 = (M')2$, $\beta^2 = k_2 - 1$ and $h(\beta r) = (a + a_1)\xi\, \beta r$

Lorentz Transformation. In the non-magnetic case, the governing equation for the pressure perturbation is a wave equation, which is invariant under a Lorentz transformation which transforms the shock path to a zero value for the appropriate variable and facilitates the transform analysis. The same transformation is applied in the present problem in order to simplify the shock path representation; however, the governing system is not invariant under the transformation. Thus, new independent variables are introduced by

$$z = \frac{x - a\xi}{(1-a^2)^{1/2}}, \quad \eta = \frac{\xi - ax}{(1 - a^2)^{1/2}}, \quad r = r \quad (5.1)$$

Using a circumflex to denote functions of the new variable, Eqs. (3.16)-(3.17) are transformed into

$$\hat{P}_{\eta\eta} = \hat{P}_{zz} + \hat{P}_{rr} + \frac{\hat{P}}{r} + m^2[\hat{B}_{rr} + \frac{\hat{B}_r}{r}] +$$

$$\frac{m^2}{1-a^2}[\hat{B}_{zz} - 2a\,\hat{B}_{z\eta} + a^2\,\hat{B}_{\eta\eta}] \quad (5.2)$$

$$[\frac{a^2 - m^2}{1-a^2}]\,\hat{B}_{zz} + \frac{2\,a(m^2-1)}{1-a^2}\,\hat{B}_{z\eta} +$$

$$\frac{(1-a^2m^2)}{1-a^2}\,\hat{B}_{\eta\eta} = \hat{P}_{rr} + \frac{\hat{P}_r}{r} + m^2\,[\hat{B}_{rr} + \frac{\hat{B}_r}{r}] \quad (5.3)$$

while the initial and boundary conditions for this system become

$$\eta \le 0,\ \hat{P} = \hat{P}_\eta = 0,\ \hat{B} = \hat{B}_\eta = 0.$$

$\eta > 0$, $z < 0$, as $z \to -\infty$, $r \to \infty$, (5.4)

$\hat{B}$, $\hat{P}$ and their derivatives vanish.

For $\eta > 0$, $z < 0$, as $r \to 0$

$$r\,\hat{P}_r + m^2\, r\,\hat{B}_r \to -a_1^2\, F[a_2(\eta + a_3 z)] \quad (5.5)$$

where $a_2 = (a+a_1)/(1-a^2)^{1/2}$, $a_3 = (1+a\,a_1)/(a+a_1)$

In terms of thenew independent variables, Eqs. (3.9), (3.11), (3,14) yield the differential relation

$$\hat{P}_{z\eta} + \hat{D}_{\eta\eta} + \frac{a}{(1-a^2)^{1/2}}\,[\frac{\partial}{\partial r} + \frac{1}{r}]\,\frac{\partial\hat{Q}}{\partial\eta} = 0 \quad (5.6)$$

Substitution of the shock conditions (4.7)-(4.8) into (5.6) gives the following differential relation valid at the shock, z = 0

$$\hat{P}_{z\eta} + \frac{1}{L_{51}}\,\hat{P}_{\eta\eta} + \frac{a}{(1-a^2)\,L_{54}}\,(\hat{P}_{rr} + \frac{P_r}{r}) =$$

$$\frac{1}{L_{51}}\,\frac{\partial^2}{\partial\eta^2}\,(L_{52}\,u^* + L_{53}\,p^*) +$$

$$\frac{a}{(1-a^2)\,L_{54}}\,(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r})(L_{21}\,u^* + L_{22}\,p^*)$$

$$-\frac{a\,L_{41}}{(1-a^2)^{1/2}}\,(\frac{\partial}{\partial r} + \frac{1}{r})\,\frac{\partial q^*}{\partial\eta} \quad (5.7)$$

Transforming thepertubation quantities u*, q* and p* [Eqs. (4.13)-(4.15)] into the new variablesand substituting into Eq. (5.7) gives the following second-order differential condition for $\hat{P}$ valid at the shock z=0.

$$- \left(\frac{\partial^2 \hat{P}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{P}}{\partial r}\right) + 2a \frac{\partial^2 \hat{P}}{\partial z\, \partial \eta} + \left(1 + \frac{1}{M^2}\right) \frac{\partial^2 \hat{P}}{\partial \eta^2} =$$

$$L_{55} \frac{\partial^2}{\partial \eta^2} \int_0^{\eta - a_4 r} \frac{F[a_2\, \theta]\, d\theta}{[(\eta-\theta)^2 - a_4^2\, r^2]^{1/2}} \tag{5.8}$$

where $a_4 = \beta/a_2$ and $L_{55} = -(1 + M^{-2})(L_{52}\, k_1 - L_{53}\, k_2) + (L_{21}\, k_1 - L_{22}\, k_2)\, a_4^2 - (1 - a^2)^{1/2} L_{54}\, L_{41}\, k_1\, a_2\, a_4^2$.

From Eq. (4.11), it follows that at z = 0

$$r\, \hat{P}_r \to L_{56}\, F[a_2 \eta] \text{ as } r \to 0 \tag{5.9}$$

where $L_{56} = (1-a^2)^{1/2} L_{54}(a_1 - L_{41}\, k_1)\, a_2 + (L_{21}\, k_1 - L_{22}\, k_2)$.

Finally, continuity of the component of the magnitic field normal to the shock gives

$$\hat{B} = 0 \text{ at } z = 0 \tag{5.10}$$

Thus, the problem is to solve the system of Eqs. (5.2)-(5.3) subject to the subsidiary conditions given by Eqs. (5.4), (5.8)-(5.10).

Analytic Solution. Equations (5.2)-(5.3), togther with the associated subsidiary conditions, will be studied by the successive application of Laplace and Hankel transforms. Thus defining

$$\mathcal{L}[\hat{P}(z,r,\eta)] = \int_0^\infty \hat{P}(z,r,\eta) \exp[-s\eta]\, d\eta = R_1\, (z,r,s)$$

$$\mathcal{H}[R_1(z,r,s)] = \int_0^\infty r\, R_1(z,r,s)\, J_0(\alpha r)\, dr = T_1(z,\alpha,s)$$

$$\mathcal{L}[\hat{B}] = R_2(z,r,s),\ \mathcal{H}[R_2] = T_2(z,\alpha,s),$$

the Laplace transforms of Eqs. (5.2)-(5.3), together with the initial conditions (5.4), give

$$s^2 R_1 - R_{1zz} = R_{1rr} + \frac{R_{1r}}{r} + m^2 [R_{2rr} + \frac{R_{2r}}{r}] + \frac{m^2}{1-a^2} [R_{2zz} - 2asR_{2z} + a^2 s^2 R_2] \tag{6.1}$$

$$[\frac{a^2 - m^2}{1-a^2}] R_{2zz} + \frac{2a(m^2-1)\, sR_{2z}}{1-a^2} + [\frac{1-m^2 a^2}{1-a^2}]\, s^2\, R_2 = R_{1rr} + \frac{R_{1r}}{r} + m^2 [R_{2rr} + \frac{R_{2r}}{r}] \tag{6.2}$$

The transforms of Eqs (5.4), (5.5), (5.8) - (5.10) give

$$z < 0, \text{ as } z \to -\infty,\ r \to \infty, \tag{6.3}$$

R_1, R_2 and their derivatives vanish.

$z < 0$, as $r \to 0$,

$$r\, R_{1r} + m^2 r\, R_{2r} = -a_1^2\, e^{s a_3 z}\, G(s) \tag{6.4}$$

$$- [\frac{\partial^2 R_1}{\partial r^2} + \frac{1}{r}\frac{\partial R_1}{\partial r}] + 2as\, \frac{\partial R_1}{\partial z} + (1 + M^{-2}) s^2 R_1 = L_{55}\, s^2 G(s)\, K_0(r\, s\, a_4) \tag{6.5}$$

$$z = 0, \text{ as } r \to 0,\ r\, \frac{\partial R_1}{\partial r} \to L_{56}\, G(s) \tag{6.6}$$

$$z = 0,\ R_2(z,\, r,\, s) = 0 \tag{6.7}$$

where $G(s) = \mathcal{L}[F(a_2 \eta)]$ and $K_0(r\, s\, a_4)$ is the modified Bessel function of the second kind.

Application of the Hankel transform to Eqs. (6.1)-(6.2), together with the conditions (6.3)-(6.4), gives.

$$[\frac{d^2}{dz^2} - (s^2 + \alpha^2)]\, T_1 + \frac{m^2}{1-a^2} [\frac{d^2}{dz^2} - 2as\, \frac{d}{dz} + a^2 s^2 - (1-a^2)\, \alpha^2]\, T_2 = -\, a_1^2\, e^{s a_3 z}\, G(s) \tag{6.8}$$

$$[(\frac{a^2 - m^2}{1-a^2})\frac{d^2}{dz^2} + \frac{2\, a(m^2-1)s}{1-a^2}\, \frac{d}{dz} + \frac{(1-m^2 a^2)s^2}{1-a^2} + m^2 \alpha^2]\, T_2 + \alpha^2\, T_1 = a_1^2\, e^{s a_3 z}\, G(s) \tag{6.9}$$

Equation (6.8)-(6.9) from a coupled system of equation for T_1 and T_2 which may be combined into a signle fourth-order differential equation with literal coefficients, constant with respect to z.

Although the general solution for this resultant equation can be given, the coefficients of the fourth-order algebraic equation, the roots of which yield the complementary integral, depend on four parameters, vis, two numerical constants measuring the shock strength and magnetic field strength and the transform parameters. Since the algebraic representation of these roots is so complicated, one specific case has been examined in detail in Appendix A for the choice of m = 1, $a^2 = 1/2$. Further, while it should be noted that the choice a = m would give a third-order equation, this choice does not not seem to lead to any algebraic simplications.

From Appendix A, the general solution is given by

$$T_1 = F_4(\alpha,s)\, L_{61}\, e^{s\lambda_1 z} + F_5(\alpha,s)\, L_{62}\, e^{s\lambda_2 z} + F_3(\alpha,s)\, e^{sa_3 z}\, G(s) \quad (6.10)$$

$$T_2 = L_{61}\, e^{s\lambda_1 z} + L_{62}\, e^{s\lambda_2 z} + F_2(\alpha,s)\, e^{sa_3 z}\, G(s) \quad (6.11)$$

The subsidiary conditions for these solutions are obtained by applying the Hankel transform to Eqs. (6.6)-(6.7) with the result that on the shock, z = 0

$$T_2 = 0 \quad (6.12)$$

and

$$2^{1/2} s \frac{d\,T_1}{dz} + (\alpha^2 + s^2 + s^2 M^{-2})\, T = \left[\frac{s^2(L_{55} - L_{56}\, a_4^2) - L_{56}\, \alpha^2}{\alpha^2 + a_4^2 s^2}\right] G(s)$$

which may be written as

$$\frac{d\,T_1}{dz} + F_6(\alpha,s)\, T_1 = F_7\,(\alpha,s)\, G(s) \quad (6.13)$$

with a new definition of coefficients. Eqs. (6.12)-(6.13) serve to evaluate L_{61} and L_{62} with the result

$$F_8 L_{61} = [F_3(s\, a_3 + F_6) - F_2 F_5(s\lambda_2 + F_6) - F_7]\, G(s) \quad (6.14)$$

$$F_8 L_{62} = [F_3(s\, a_3 + F_6) + F_2 F_4(s\lambda_1 + F_6) + F_7]\, G(s) \quad (6.15)$$

where $F_8 = F_5(s\lambda_2 + F_6) - F_4(s\lambda_1 + F_6)$.

Thus, the solution for the transforms is given by Eqs. (6.10)-(6.11) and (6.14)-(6.15). A formal inversion of these tranforms gives the solution in terms of repeated infinite intergrals. Solutions for the other flow parameters may then be obtained from the governing differntial equations and the subsidiary conditions.

Part II Plane Flow

The Basic Equations. With respect to an (x,y)-plane coordinate system, the unsteady flow equations with velocity vector (u,v), subjected to an applied magnetic field (B_1, B_2), may be written as

$$\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0 \quad (7.1)$$

$$\rho(u_t + uu_x + vu_y) + p_x = -B_2(B_{2x} - B_{1y})/\mu \quad (7.2)$$

$$\rho(v_t + uv_x + vv_y) + p_y = B_1(B_{2x} - B_{1y})/\mu \quad (7.3)$$

$$B_{1t} + v\, B_{1y} + B_1 v_y - B_2 u_y - u\, B_{2y} = 0 \quad (7.4)$$

$$B_{2t} + u\, B_{2x} + B_2 u_x - B_1 v_x - v\, B_{1x} = 0 \quad (7.5)$$

$$B_{1x} + B_{2y} = 0 \quad (7.6)$$

$$s_t + u\, s_x + v\, s_y = 0 \quad (7.7)$$

No component of the magnetic field normal to the plane of the flow is included, for an analysis similar to that carried out in Part I shows that any perturbation of this component will vanish identically, provided the component is initially zero.

Aligned Fields. When Eqs. (7.1) – (7.7) are linearized in the neighborhood of (u,v) = (0,0), $\vec{B} = (B_1, 0)$, relative to a coordinate system fixed in the undisturbed flow behind the plane shock, the following system of linear euqations is obtained

$$\overline{\rho_t} + \rho[\overline{u_x} + \overline{v_y}] = 0 \quad (8.1)$$

$$\rho\overline{u}_t + p_x = 0 \quad (8.2)$$

$$\rho\overline{v}_t + \overline{p_y} = B_1\,(\overline{B_{2x}} - \overline{B_{1y}})\,\mu^{-1} \quad (8.3)$$

$$\overline{B_{1t}} - B_1\overline{v_y} = 0 \quad (8.4)$$

$$\overline{B_{2t}} - B_1\overline{v_x} = 0 \quad (8.5)$$

$$\overline{B_{1x}} + \overline{B_{2y}} = 0 \quad (8.6)$$

$$\overline{s_t} = 0 \text{ or } \overline{p_t} - c^2\overline{\rho_t} = 0 \quad (8.7)$$

Using the same definitions of symbols as in section 3, save for now letting $B = \overline{B_1}/B_1$, $Q = \overline{v}/c$ and carrying out a similar reduction gives the following system of coupled linear equations

$$P_{\xi\xi} = P_{xx} + P_{yy} + m^2(B_{yy} + B_{xx}) \qquad (8.8)$$

$$B_{\xi\xi} = P_{yy} + m^2 (B_{yy} + B_{xx}) \qquad (8.9)$$

where $m^2 = b_1^2/c^2$

Since the flow is assumed always super-true-sonic, the flow patterns on the two sides of a two-dimensional airfoil will be independent of each other, so that it suffices to consider the solution for $y > 0$.

The initial and boundary conditions are that for

$$\xi \le 0,\ P = P_\xi = 0,\ B = B_\xi = 0 \qquad (8.10)$$

and the shock conditions given by Eqs. (4.1) – (4.5) hold with r replaced by y and Q now denotes $\overline{v}/c$ and $q^* = \overline{v}^2/V$. These give

$$D = (P - L_{52}\, u^* - L_{53}\, p^*)/L_{51} \qquad (8.11)$$

$$Q_\xi = L_{41}\, q^*_\xi + (P_y - L_{21}\, u^*_y - L_{22}\, p^*_y)/L_{54} \qquad (8.12)$$

With the upper surface of the airfoil represented by the function f, the boundary condition behind the shock may be written as

$$Q = a_1\, f'(x + a_1\xi) \qquad (8.13)$$

on the surface of the airfoil.

From the second momentum equation, (8.3), it follows that

$$P_y + m^2\, B_y = -a_1^{\,2}\, f''(x + a_1\xi) \qquad (8.14)$$

on $y = 0$. On the shock along the airfoil, $Q = a_1\, \acute{f}\ [a + a_1)\xi]$ which, together with Eq. (8.12), gives

$$P_y = L_{54}\, a_1(a + a_1)\, f''[(a + a_1)\xi] + L_{21}\, u^*_y + L_{22}\, p^*_y - L_{54}\, L_{41}\, q^*_\xi \qquad (8.15)$$

At infinity, as

$x \to -\infty$, $y \to \infty$, P, B and their derivatives vanish. Finally, the distrubance field ahead of the shock may be expressed as [3]

$$u^* = \overline{u}_2/V = -\, k_1\, f'\ [x + a_1\, \xi - \beta y]/\beta$$

$$p^* = \overline{p}_2/\gamma p_0 = -\, k_2\, u^*/k_1$$

$$q^* = \overline{v}_2/V = -\, \beta\, u^* \qquad (8.16)$$

Introduction of the Lorentz transformation (5.1) transforms the system of Eqs. (8.8) – (8.9) to

$$\hat{P}_{\eta\eta} = \hat{P}_{zz} + \hat{P}_{yy} + m^2\, \hat{B}_{yy} + \frac{m^2}{1-a^2}\, [\hat{B}_{zz} - 2a\, \hat{B}_{z\eta} + a^2\, \hat{B}_{\eta\eta}] \qquad (8.17)$$

$$\left[\frac{a^2-m^2}{1-a^2}\right] \hat{B}_{zz} + \frac{2a(m^2-1)}{1-a^2}\, \hat{B}_{z\eta} + \frac{(1-m^2a^2)}{1-a^2}\, \hat{B}_{\eta\eta} = \hat{P}_{yy} + m^2\, \hat{B}_{yy} \qquad (8.18)$$

The initial conditions are $\hat{P} = \hat{P}_\eta = 0$, $\hat{B} = \hat{B}_\eta = 0$ for $\eta \leq 0$, while the boundary contitions are $y \to \infty$, $x \to -\infty$, $\hat{B}$, $\hat{P}$ and their derivatives vanish and at $y = 0$

$$\frac{\partial \hat{P}}{\partial y} = -\, a_1^{\,2}\, f''\ [a_2(\eta + a_3 z)] \qquad (8.19)$$

A combination, entirely similar to that carried out in Part I, gives the differential relation

$$-\frac{\partial^2 \hat{P}}{\partial y^2} + 2a\, \frac{\partial^2 \hat{P}}{\partial z \partial \eta} + (1 + M^{-2})\, \frac{\partial^2 \hat{P}}{\partial \eta^2} = \frac{L_{55}}{a^4}\, \frac{\partial}{\partial \eta}\, f''\ [a_2(\eta - a_4 y)] \qquad (8.20)$$

valid on the shock. As before,

$$\hat{B} = 0 \qquad (8.21)$$

on the shock. Finally, at $z = 0$, $y = 0$

$$\frac{\partial \hat{P}}{\partial y} = L_{56}\, f''\ [a_2\eta] \qquad (8.22)$$

The system of Eqs. (8.17)-(8.18) plus the subsidiary conditions will be solved by applying the Laplace transform with respect to η and the Fourier cosine tranform with respect to y, i.e., with $\mathcal{L}(\hat{p}) = R_1$,

$$\mathcal{F}_c[R_1(z,y,s)] = \int_0^\infty R_1(z,y,s)\cos\alpha y\, dy =$$

$$T_1(z,y,s) \tag{8.23}$$

Then, with $\mathcal{L}(\hat{B}) = R_2, \mathcal{F}_c(R_2) = T_2$, the tramsforms of Eqs. (8.17)-(8.18), together with the subsidiary conditions, give

$$\left[\frac{d^2}{dz^2} - (s^2+\alpha^2)\right]T_1 + \frac{m^2}{1-a^2}\left[\frac{d^2}{dz^2} - 2as\frac{d}{dz} + a^2s^2\right.$$

$$\left. - (1-a^2)\alpha^2\right]T_2 = -a_1^2 e^{sa_3 z} G(s) \tag{8.24}$$

$$\left[\frac{(a^2-m^2)}{1-a^2}\frac{d^2}{dz^2} + \frac{2a(m^2-1)s}{1-a^2}\frac{d}{dz} + \frac{(1-m^2a^2)s^2}{1-a^2}\right.$$

$$\left. + m^2\alpha^2\right]T_2 + \alpha^2 T_1 = a_1^2 e^{sa_3 z} G(s) \tag{8.25}$$

Structurally, this formulation is identical to that obtained for axi-symmetric flow. Thus, the solution presented in section 6 for the special choices of the numerical parameters also is a solution of the system of Eqs. (8.24)-(8.25) and the subsidiary conditions. The only difference is that α is now the parameter from the Fourier cosine transform. A formal inversion gives the solution in terms of repeated infinite integrals.

Transverse Magnetic Field. When Eqs. (7.1)-(7.7) are linearized in the neighbourhood of (u,v) = (0,0), (0, B_2), the following system of linear equations is obtained

$$\overline{\rho_t} + \rho(\overline{u_x} + \overline{v_y}) = 0 \tag{9.1}$$

$$\overline{u_t} + \frac{\overline{p_x}}{\rho} = -b_2^2\frac{\overline{B_{2x}}}{B_2} + b_2^2\frac{\overline{B_{1y}}}{B_2} \tag{9.2}$$

$$\rho\overline{v_t} + \overline{p_y} = 0 \tag{9.3}$$

$$\overline{B_{1t}} - B_2\overline{u_y} = 0 \tag{9.4}$$

$$\overline{B_{2t}} + B_2\overline{u_x} = 0 \tag{9.5}$$

$$\overline{B_{1x}} + \overline{B_{2y}} = 0 \tag{9.6}$$

$$\overline{p_t} - c^2\overline{\rho_t} = 0 \tag{9.7}$$

Using the same definition of symbols as in section 8, except for letting $\bar{B} = \bar{B}_2/B_2$, $m^2 = b_2^2/c^2$. the peturbation equations may be rewritten as

$$A_\xi + D_x + Q_y = 0 \tag{9.8}$$

$$D_\xi + P_x = m^2\left(\frac{\overline{B_{1y}}}{B_2} - B_x\right) \tag{9.9}$$

$$Q_\xi + P_y = 0 \tag{9.10}$$

$$\frac{\overline{B_{1\xi}}}{B_2} - D_y = 0 \tag{9.11}$$

$$B_\xi + D_x = 0 \tag{9.12}$$

$$P_\xi - A_\xi = 0 \tag{9.13}$$

$$B_y + \frac{\overline{B_{1x}}}{B_2} = 0 \tag{9.14}$$

These equations may be combined to give the following system of coupled linear equations.

$$P_{\xi\xi} = P_{xx} + P_{yy} + m^2[B_{xx} + B_{yy}] \tag{9.15}$$

$$B_{\xi\xi} = P_{xx} + m^2[B_{xx} + B_{yy}] \tag{9.16}$$

For this problem, it is necessary to utlize the perturbed form of the transition relations across a normal magnetohydrodynamic shock. A derivation of these relations in a form particularly well-suited for the present problem is given in Appendix C. These relations may be written as

$$A = E_3\psi_\xi + E_4M_2u^* + E_6 B^* + E_7 p^* \tag{9.17}$$

$$B = E_3\psi_\xi + E_4M_2u^* + (E_6+1)B^* + E_5 p^* \tag{9.18}$$

$$P = E_8\psi_\xi + E_9M_2u^* + E_{10} B^* + E_{11} p^* \tag{9.19}$$

$$D = E_{12}\psi_\xi + E_{13}M_2u^* + E_{14} B^* + E_{15} p^* \tag{9.20}$$

$$Q = L_{41} q^* + L_{42}\psi_y \tag{9.21}$$

Where Eq. (9.21) is the same as Eq. (4.4) with r replaced by y and $p^* = \bar{p}_2/\gamma p_2$; the other symbols are the same as defined previously. These relations connect the quantities in the region in front of the shock (subscript 2) to those in the region behind the shock (subscript 1). In addition, since the normal component of the magnetic field must be continuous across the shock, there is the further condition that the perturbed x-component of the induction is continous across the shock, i.e., $(\overline{B_1})_1 = (\overline{B_1})_2$. These conditions will be applied at $x = a\xi$.

The initial conditions are that for

$$\xi \leq 0, \; P = P_\xi = 0, \; B = B_\xi = 0 \tag{9.22}$$

Elimination of $\psi\xi$ from Eqs. (9.19)-(9.20) and (9.18)-(9.19) and ψ from (9.19) and (9.21) gives

$$D = \frac{E_{12}}{E_8}\left[P - \left(E_9 - \frac{E_{13}E_8}{E_{12}}\right) u^* - \left(E_{10} - \frac{E_{14}E_8}{E_{12}}\right) B^* - \left(E_{11} - \frac{E_{15}E_8}{E_{12}}\right) p^*\right] \equiv$$

$$[P - E_{21} u^* - E_{22} p^* - E_{23} B^*]/E_{20} \tag{9.23}$$

$$B = \frac{E_3}{E_8}\left[P - \left(E_9 - \frac{E_4E_8}{E_3}\right) M_2 u^* - \left(E_{10} - (1 + E_6)\frac{E_8}{E_3}\right) B^* - \left(E_{11} - E_5 \frac{E_8}{E_3}\right) p^*\right]$$

$$\equiv [P - E_{25} u^* - E_{26} p^* - E_{27} B^*]/E_{24} \tag{9.24}$$

$$Q_\xi = [P_y - E_9M_2 u^*_y - E_{10}B^*_y - E_{11}P_y^*]/E_{28} + L_{41} q^*_\xi \tag{9.25}$$

where $E_{28} = E_8/L_{42}$.

The perturbed flow ahead of the shock (region 1) has been determined previously[3], and, as in the other cases, it was shown that the magnetic field was not perturbed in the insulator (airfoil, in the present context); thus, continuity of the magnetic field across the interface shows that all magnetic field components (and x-derivatives which are formed by differentiating along the surface of the airfoil, i.e., y=0 in the linearized formulation) must vanish on the surface of the airfoil in the fluid. Further, it is easy to see that, within the limits of the linearized analysis, the magnetic field is not perturbed throughout the interior of the airfoil.

As in the previous section, the boundary conditions on the surface of the airfoil behind the shock are

$$Q = a_1 f' (x + a_1 \xi) \tag{9.26}$$

$$P_y = - a_1^2 f'' (x + a_1 \xi) \tag{9.27}$$

and, since $B_y = - \overline{B_{1x}}/B_2$,

$$B_y = 0 \text{ on } y = 0 \tag{9.28}$$

Also, along the shock on the surface of the airfoil

$$Q = a_1 f' [(a + a_1) \xi] \tag{9.29}$$

which, together with Eq. (9.25), gives

$$P_y = E_{28} a_1(a + a_1) f'' [(a + a_1) \xi] + M_2E_9 u^*_y + E_{10}B^*_y + E_{11} P^*_y - E_{28}E_{41} q^*_\xi \tag{9.30}$$

At infinity, as $x \to \infty$, $y \propto$], P, B and their derivatives are to vanish. Finally, the disturbance field ahead of the shock (region 2) may be expressed as[3]

$$u^* = \frac{\overline{u_2}}{V} = \frac{\omega n_2^3}{(a-n_2)(n_2^2-m^2)} \{ \delta(y) f' (x + a_1 \xi) - f' (x + a_1 \xi - y/\omega)\} \tag{9.31}$$

$$q^* = \frac{\overline{v_2}}{c_2} = n_2 f' [x + a_1 \xi - y/\omega]$$

$$p^* = \frac{\overline{p_2}}{\gamma p_2} = n_2^2 \omega f' [x + a_1 \xi - y/\omega]$$

$$B^* = \frac{\overline{B_2}}{B_2} = \frac{(n_2-a)}{n_2} u^*$$

where $n_2 = u_2/c_2$, $\omega^2 = (n_2^2-m^2)/n^2(n^2-m^2-1)$.

Introduction of the Lorentz transformation Eq. (5.1), trnasforms the system of Eqs. (9.15)-(9.16) into

$$\hat{P}_{\eta\eta} = \hat{P}_{zz} + \hat{P}_{yy} + m^2 \hat{B}_{yy} + \frac{m^2}{1-a^2} [\hat{B}_{zz} - 2a \hat{B}_{z\eta} + a^2 \hat{B}_{\eta\eta}] \tag{9.32}$$

$$\frac{(a^2-m^2)}{1-a^2} \hat{B}_{zz} + \frac{2a(m^2-1)}{1-a^2} \hat{B}_{z\eta} + \frac{(1-m^2a^2)}{1-a^2} \hat{B}_{\eta\eta} = \frac{\hat{P}_{zz}}{1-a^2} - \frac{2a \hat{P}_{z\eta}}{1-a^2}$$

$$+ \frac{a^2 \hat{P}_{\eta\eta}}{1-a^2} + m^2 \hat{B}_{yy} \qquad (9.33)$$

In the transformed variables, the initial conditions are

$$\hat{P} = \hat{P}_\eta = 0,\ \hat{B} = \hat{B}_\eta = 0 \text{ for } \eta < 0,$$

while the boundary conditions are that as $y \to \infty$, $x \to -\infty$, $\hat{B}$, $\hat{P}$ and their derivatives vanish; at $y = 0$

$$\frac{\partial \hat{P}}{\partial y} = - a_1^2 f'' [a_2(\eta + a_3 z)],\ \frac{\partial \hat{B}}{\partial y} = 0 \qquad (9.34)$$

Two relations valid at the shock may be obtained quite easily. From Eqs. (9.8), (9.12)-(9.13), $P_\xi - B_\xi + Q_y = o$, or

$$\frac{\partial}{\partial \eta} (\frac{\partial}{\partial \eta} - a \frac{\partial}{\partial z})(\hat{P} - \hat{B}) + \hat{Q}_y (1-a^2)^{1/2} = 0 \qquad (9.35)$$

and Eq. (9.24).

Substitution of the appropriate quantities into (9.35) and (9.24) gives, entirely similar to the procedure of the previous sections,

$$\frac{\partial}{\partial \eta} [\frac{\partial}{\partial \eta} - a \frac{\partial}{\partial z}] (\hat{P} - \hat{B}) + \frac{\hat{P}_{yy}}{E_{28}} =$$

$$\frac{1}{E_{28} a_2 \omega} \frac{\partial}{\partial \eta} \{E_{11} n_2^2 + \frac{n_2^2}{n_2^2 - m^2} (E_{10} - \frac{n_2 M_2 E_9}{a - n_2})$$

$$+ L_{41} (1-a^2)^{1/2} E_{28}] f''[a_2(\eta - a_5 y)] +$$

$$(\frac{n_2 M_2 E_9}{a - n_2} - E_{10}) \delta/y) \frac{f'' (a_2\eta) n_2^2}{n_2^2 - m^2} \equiv$$

$$E_{30} \frac{\partial}{\partial \eta} f'' [a_2(\eta - a_5 y)] + E_{31} \delta(y) \frac{\partial}{\partial \eta} f'' [a_2 \eta] \qquad (9.36)$$

$$\hat{B} - \frac{\hat{P}}{E_{24}} =$$

$$\{[\frac{E_{25}}{E_{24}} + \frac{E_{27}(n_2 - a_1)}{E_{24} n_2}] \frac{\omega n_2^3}{(a_2 - n_2)(n_2^2 - m^2)} - \frac{E_{26}}{E_{24}} n_2 \omega\}$$

$$f' [a_2(\eta - a_5 y)]$$

$$-[\frac{E_{25}}{E_{24}} + \frac{E_{27}}{E_{24}} \frac{(n_2 - a_1)}{n_2}] [\frac{\omega n_2^3}{(a - n_2)(n_2^2 - m^2)}] \delta(y) f' [a_2 \eta]$$

$$\equiv E_{32} f' [a_2(\eta - a_5 y)] + E_{33} \delta(y) f' [a_2 \eta] \qquad (9.37)$$

where $a_5 = 1/a_2\ \omega$. Finally, on the shock at the airfoil, $y = 0$,

$$P_y = [E_{28} a_1(a+a_1) - n_2^2 E_{11}$$

$$- E_{28} E_{41} (1 - a^2)^{1/2} a_2 n_2] f'' [a_2 \eta] \equiv E_{34} f'' [a_2 \eta] \qquad (9.38)$$

$$B_y = 0$$

The system of Eqs. (9.32)-(9.33) and the associated subsidiary conditions will be studied by the successive application of Laplace and Fourier cosine transforms. Thus, using essentially the same notation as in section 8, the Lapalxce transforms of Eqs. (9.32)-(9.33), together with the initial conditions, give

$$R_{1zz} - s^2 R_1 + R_{1yy} + m^2 R_{2yy} +$$

$$\frac{m^2}{1-a^2} [R_{2zz} - 2as R_{2z} + a^2 s^2 R_2] = 0 \qquad (9.39)$$

$$(a^2 - m^2) R_{2zz} + 2a(m^2-1) s R_{2z} +$$

$$(1-m^2 a^2) s^2 R_2 - (1-a^2) m^2 R_{2yy} =$$

$$R_{1zz} - 2as R_{1z} + a^2 s^2 R_1 \qquad (9.40)$$

while the transforms of Eq. (9.34) give

$$R_{1y} = -a_1^2 \mathcal{L}\{f'' [a_2(\eta + a_3 z)]\} \equiv$$

$$-a_1^2 e^{s a_3 z} G_1(s),\ R_{2y} = 0 \qquad (9.41)$$

where $G_1(s) = \mathcal{L}[F''(a_2\eta)]$

Application of the Fourier cosine transform to Eqs. (9.39)-(9.40), together with Eq.(9.41), gives

$$[\frac{d^2}{dz^2} - (s^2 + \alpha^2)]T_1 + \frac{m^2}{1-a^2} [\frac{d^2}{dz^2} - 2as \frac{d}{dz} +$$

$$a^2 s^2 - m^2 \alpha^2 (1 - a^2)]T_2 = - a_1^2 e^{s a_3 z} G_1(s) \qquad (9.42)$$

$$[(a^2 - m^2) \frac{d^2}{dz^2} + 2as(m^2-1) \frac{d}{dz} + (1 - m^2 a^2) s^2$$

$$+ m^2(1 - a^2)\ \alpha^2]T_2 = [\frac{d^2}{dz^2} - 2as\frac{d}{dz} + a^2s^2]T_1 \quad (9.43)$$

A detailed analysis of this system will be made for the special choices of m = 1, $a^2 = 1/2$. From Appendix B, the general solution for this special case may be written as

$$T_1 = F_9(\alpha, s)\ L_{71}\ e^{\lambda_5 z} + F_{10}(\alpha, s)\ L_{72}\ e^{\lambda_6 z} + \Omega_1(\alpha, s)\ e^{sa_3 z} \quad (9.44)$$

$$T_2 = L_{71}\ e^{\lambda_5 z} + L_{72}\ e^{\lambda_6 z} + \Omega_2(\alpha, s)\ e^{sa_3 z} \quad (9.45)$$

The coefficients L_{71} and L_{72} may be determined from the transforms of Eqs. (9.36)-(9.38); application of the Laplace transform gives

$$\frac{1}{E_{28}}\frac{\partial^2 R_1}{\partial y^2} + s^2 R_1 - as\frac{\partial R_1}{\partial z} - s^2 R_2 + a\ s\frac{\partial R_2}{\partial z} = [E_{30}\ e^{-a_5 sy} + E_{31}\ \delta(y)]\ s\ G_1(s) \quad (9.46)$$

$$R_2 - E_{24}^{-1}R_1 = [E_{32}\ e^{-a_5 sy} + E_{33}\ \delta(y)]\ G_2(s) \quad (9.47)$$

where $G_2(s) = \mathcal{L}\,[f'(a_2\eta)]$.

$$\frac{\partial R_1}{\partial y} = E_{34}\ G_1(s), \quad \frac{\partial R_2}{\partial y} = 0 \quad (9.48)$$

Next, application of the Fourier cosine transform gives

$$(s^2 - \frac{\alpha^2}{E_{28}})T_1 - a\ s\frac{dT_1}{dz} + a\ s\frac{dT_2}{dz} - s^2T_2 = [E_{34} + \frac{s^2 a_5 E_{30}}{\alpha^2 + a_5^2 s^2} + s\ E_{31}]\ G_1(s) \equiv \Omega_3(\alpha, s) \quad (9.49)$$

$$T_2 - \frac{T_1}{E_{24}} = [E_{33} + \frac{E_{32} s\ a_5}{\alpha^2 + s^2 a_5^2}]G_2(s) \equiv \Omega_4(\alpha, s) \quad (9.50)$$

Substitution of Eqs. (9.44)-(9.45) into (9.49)-(9.50) and letting $a^2 = 1/2$ gives

$$[F_9(s^2 - s\lambda_5\ 2^{1/2} - \frac{\alpha^2}{E_{28}}) + s\ \lambda_5\ 2^{1/2}]\ L_{71} + [F_{10}(s^2 - s\lambda_6\ 2^{1/2} - \frac{\alpha^2}{E_{28}}) + s\lambda_6 2^{1/2} - s^2]\ L_{72}$$
$$= \Omega_3 + (\frac{\alpha^2}{E_{28}} - s^2 + a_3 s^2\ 2^{1/2})\ \Omega_1 + (s^2 - a_3\ 2^{1/2})\ \Omega_2 \quad (9.51)$$

$$(1 - \frac{F_9}{E_{24}})L_{71} + (1 - \frac{F_{10}}{E_{24}})\ L_{72} = \Omega_4 - \Omega_2 - \frac{\Omega_1}{E_{24}} \quad (9.52)$$

Solving this simple simultaneous system gives L_{71}, L_{72}, which may be introduced into Eqs. (9.44)-(9.45) to give the final solution for the transforms. A formal inversion of the transforms gives the solution for arbitrary profiles in terms of repeated infinite integrals. The other flow parameters and the path of the diffracted shock may be determined quite easily.

REFERENCES

1. N.L. Arora, J. Fluid Mech., **34**, 209 (1968).
2. R.M. Gundersen, *Linearized Analysis of One-dimensional Magnetohydrodynamic Flows* (Springer-Verlag, Berlin, (1964).
3. R.M. Gundersen, Z.angew.Math, Phys., **19**, 412 (1968).
4. R.M. Gundersen, Z.angew. Math. Phys., **20**, 343 (1969).
5. R.M. Gundersen, Ann.Soc.Sci. Bruxelles, **83**, 235 (1969).
6. R.M. Gundersen, Z. angew. Math. Phys, **19**, 316 (1968).
7. N. Conkwright, *Introduction to the Theory of Equations* (Ginn and Company, New York, 1941).

APPENDIX A

Quartic Equation for Aligned Fields

In appendices A and B, only the final results are used in the body of the paper; thus, symbols which may have been used for other quantities in the paper are utilized.

The algebraic equation, whose roots yield the solution for the homogenous system associated with Eqs. (6.8)-(6.9) for the choice $a^2 = 1/2$ is

$$\lambda^4 - 2s^2\lambda^2 - 2^{3/2}s\alpha^2\lambda + s^4 + 3\alpha^2 s^2 = 0 \quad (A.1)$$

Letting $\lambda = s\nu$, $\beta = \alpha^2/s^2$ gives

$$\nu^4 - 2\nu^2 - 2^{3/2}\beta\nu + 1 + 3\beta = 0$$

which will be studied by Descartes' method[7]; thus, writing (A-1) as

$$(\nu^2 + k\nu + h)(\nu^2 - k\nu + m) \quad (A.2)$$

and equating coefficients gives

$$h + m - k = -2, \quad k(m - h) = -2^{3/2}\beta,$$

$$m\,h = 1 + 3\beta \quad (A.3)$$

which gives the following cubic for k^2:

$$k^6 - 4k^4 - 12\beta k^2 - 8\beta^2 = 0 \quad \text{or with} \quad 2n = k^2$$

$$n^3 - 2n^2 - 3\beta n - \beta^2 = 0 \quad (A.4)$$

The reduced cubic may be obtained by setting $n = n_1 + 2/3$ to give

$$n_1^3 - 3(\beta + 4/9)n_1 - (\beta^2 + 2\beta + 16/27) = 0 \quad (A.5)$$

the solution of which is given by

$$n_1 = \left\{\frac{\beta^2 + 2\beta + 16/27 + \beta(\beta^2 - 4/27)^{1/2}}{2}\right\}^{1/3} +$$

$$\{\beta^2 + 2\beta + 16/27 - \beta(\beta^2 - 4/27)^{1/2}\}^{1/3} \quad (A.6)$$

Thus $k^2 = 2(n_1 + 2/3) = 2n$

$$m = \frac{k^3 - 2k + 2^{3/2}\beta}{2k}$$

$$h = \frac{k^3 - 2k + 2^{3/2}\beta}{2k}$$

The roots ν obtained from the first factor of (A.2) are unacceptable since they would lead to solutions which will not vanish at negative infinity; consequently, only the two roots obtained from the second factor need be considered. Defining

$$2\lambda_1 = k + (k^2 - 4m)^{1/2},$$

$$2\lambda_2 = k - (k^2 - 4m)^{1/2}$$

a consistent solution of the homogeneous system associated with Eqs. (6.8)-(6.9) may be written as

$$T_2 = L_{61}e^{s\lambda_1 z} + L_{62}e^{s\lambda_2 z} \quad (A.7)$$

$$T_1 = [s^2\lambda_1^2 - (s^2 + \alpha^2)]\alpha^{-2}L_{61}e^{s\lambda_1 z} +$$

$$[s^2\lambda_2^2 - (s^2 + \alpha^2)]\alpha^{-2}L_{62}e^{s\lambda_2 z}$$

$$\equiv F_4(\alpha,s)L_{61}e^{s\lambda_1 z} + F_5(\alpha,s)L_{62}e^{s\lambda_2 z} \quad (A.8)$$

where L_{61}, L_{62} are arbitrary. Finally, a particular solution of Eqs. (6.8)-(6.9) may be written as

$$T_1 = \frac{[s^2 a_3^2 - 2^{3/2}s^2 a_3 + 2s^2]\,a_1^2 e^{s a_3 z}G(s)}{(s\,a_3)^4 - 2s^2(s\,a_3)^2 - 2^{3/2}s\,\alpha^2(s\,a_3) + s^4 + 3\alpha^2 s^2} \equiv F_3(\alpha,s)e^{s a_3 z}G(s)$$

$$T_2 = \frac{s^2(a_3^2 - 1)\,a_1^2 G(s)e^{s a_3 z}}{(s\,a_3)^4 - 2s^2(s\,a_3)^2 - 2^{3/2}s\,\alpha^2(s\,a_3) + s^4 + 3\alpha^2 s^2} \equiv F_2(\alpha,s)e^{s a_3 z}G(s)$$

APPENDIX B

Quartic Equation for the Transverse Field Case

The algebraic equation for λ, whose roots give the solution for the homogeneous system associated with Eqs. (9.42)-(9.43) with $a^2 = 1/2$, may be simplified by writing $\lambda - s/2^{1/2} = s\nu$, this gives

$$3\nu^4 - 2^{3/2}\nu^3 - \nu^2 + 2^{3/2}\beta^2\nu - \beta^4 = 0 \quad (B.1)$$

where $\beta^2 = (\alpha^2/s^2) + 1/2$. The further substitution $\nu = \mu + 2^{1/2}/6$ gives an equation to which Descartes' method may be applied, namely,

$$\mu^4 - \frac{2\mu^2}{3} + \frac{2^{3/2}}{3}(\delta + \frac{1}{18})\mu - \frac{1}{3}(\delta^2 - \frac{1}{36})$$

$$= (\mu^2 + k\mu + h)(\mu^2 - k\mu + m) = 0 \quad (B.2)$$

where $\delta = B^2 - 1/3$. Equating coefficients gives $h + m - k^2 = -2/3, k(m - h) = 2^{3/2}(\delta + \frac{1}{18})/3, m h = -(\delta^2 - 1/36)/3$ which leads to the following cubic for k^2

$$k^6 - \frac{4}{3}k^4 + \frac{4k^2}{3}(\delta^2 + \frac{11}{36}) - \frac{8}{9}(\delta + \frac{1}{18})^2 = 0 \quad (B.3)$$

The reduced cubic is obtained by letting $n + 4/9 = K^2$ to give the solution

$$3n = \{(4\delta^2 + \frac{4\delta}{3} - \frac{1}{27}) + 2[16\delta^6 - \frac{8\delta^4}{3} + \frac{8\delta^2}{3} - \frac{35\delta^2}{27} - \frac{2\delta}{81} - \frac{31}{729}]^{1/2}\}^{1/3} + \{(4\delta^2 + \frac{4\delta}{3} - \frac{1}{27}) - 2[16\delta^6 - \frac{8\delta^4}{3} + \frac{8\delta^2}{3} + \frac{35\delta^2}{27} - \frac{2\delta}{81} - \frac{31}{729}]^{1/2}\}^{1/3}$$

Thus

$$k^2 = n + 4/9$$

$$m = \frac{k^3 - 2k/3 + 2^{3/2}(\delta + 1/18)/3}{2k}$$

$$h = \frac{k^3 - 2k/3 - 2^{3/2}(\delta + 1/18)/3}{2k}$$

As in Appendix A, only the roots obtained from the second factor in (B.2) are acceptable; defining

$$2\lambda_3 = k + (k^2 - 4m)^{1/2},$$

$$2\lambda_4 = k - (k^2 - 4m)^{1/2},$$

a consistent solution of the homogeneous system associated with Eqs. (9.42)-(9.43) may be written as

$$T_2 = L_{71}\exp\{[\lambda_3 + 2^{3/2}/3]\, sz\} + L_{72}\exp\{[\lambda_4 + 2^{3/2}/3]\, sz\} \equiv L_{71} e^{\lambda_5 z} + L_{72} e^{\lambda_6 z} \quad (B.4)$$

$$T_1 = [\frac{2(\lambda_3 + 2^{1/2}/6)^2 s^2 - \alpha^2}{s^2 + \alpha^2 - s^2(\lambda_3 + 2^{3/2}/3)^2}] L_{71}\exp\{[\lambda_3 + 2^{3/2}/3]\, sz\} + [\frac{2(\lambda_4 + 2^{1/2}/6)^2 s^2 - \alpha^2}{s^2 + \alpha^2 - s^2(\lambda_4 + 2^{3/2}/3)^2}] L_{72}\exp\{[\lambda_4 + 2^{3/2}/3]\, sz\}$$

$$\equiv F_9(\alpha, s) L_{71} e^{\lambda_5 z} + F_{10}(\alpha, s) L_{72} e^{\lambda_6 z} \quad (B.5)$$

where L_{71} and L_{72} are arbitrary. Finally a particular solution of (9.42)-(9.43) may be written as

$$T_1 = \frac{a_1^2(s^2 a_3^2 - s^2 - \alpha^2)\, e^{s a_3 z}\, G(s)}{2\Omega(\alpha, s)}$$

$$\equiv \Omega_1(\alpha, s)\, e^{s a_3 z}$$

$$T_2 = \frac{a_1^2 s^2 (a_3 - 2^{-1/2})^2\, e^{s a_3 z}\, G(s)}{\Omega(\alpha, s)}$$

$$\equiv \Omega_2(\alpha, s)\, e^{s a_3 z}$$

where

$$\Omega(\alpha, s) = \alpha^2 s^2 (a_3 - 2^{-1/2})^2 + 2s^4(a_3 - 2^{-1/2})^4 - [s^2 a_3^2 - s^2 - \alpha^2]^2/2$$

APPENDIX C

Perturbed Form of the MHD Shock Relations

In [2], a perturbed form of the normal magnetohydrodynamic shock conditions for flow perturbed in front of and behind the initially uniform shock was given. For the present paper, a somewhat different derivation is more appropriate. Thus, let the flow in front of and behind the shock be denoted by subscripts one and two, respectively. Let $\tau = p_2/p_1$, $\sigma = \rho_2/\rho_1$, $m = b/c$, $n = u/c$, $M = (V - u)/c$, $\theta = (\gamma + 1)/(\gamma - 1)$ where V is the shock speed.

Then, from

$$M_1^2 = \{\frac{2}{\gamma-1} + m_1^2[\frac{\gamma}{\gamma-1} + \frac{(2-\gamma)\sigma}{\gamma-1}]\}\frac{\sigma}{\theta-\sigma},$$

$$\frac{\overline{M_1}}{M_1} = \frac{E_1}{E_2}\frac{\overline{m_1}}{m_1} + [\frac{(\frac{2-\gamma}{\gamma-1})m_1^2\,\sigma(\theta-\sigma) + \theta E_2}{2(\theta-\sigma)E_2}]\frac{\overline{\sigma}}{\sigma} \quad (C.1)$$

where $E_1 = m_1^2[\frac{\gamma}{\gamma-1} + \frac{(2-\gamma)\sigma}{\gamma-1}]$, $E_2 = E_1 + \frac{2}{\gamma-1}$

Since

$$\frac{\overline{m_1}}{m_1} = \frac{\overline{B_1}}{B_1} - \frac{\gamma\,\overline{p_1}}{2\gamma p_1}, \quad (C.2)$$

$$\frac{\overline{M_1}}{M_1} = \frac{\overline{V-u_1}}{V-u_1} - \frac{\overline{c_1}}{c_1}, \quad (C.3)$$

Eq. (C.1) may be rewritten as

$$[\frac{(2-\gamma)\, m_1^2\,\sigma(\theta-\sigma)/(\gamma-1) + \theta E_2}{2(\theta-\sigma)E_2}]\frac{\overline{\sigma}}{\sigma} = \frac{\overline{V}}{M_1 c_1} - \frac{\overline{u_1}}{M_1 c_1} - \frac{E_1}{E_2}\frac{\overline{B_1}}{B_1} + [\frac{\gamma E_1}{2E_2} - \frac{(\gamma-1)}{2}]\frac{\overline{p_1}}{\gamma p_1} \text{ or}$$

$$\frac{\overline{\sigma}}{\sigma} = E_3\frac{\overline{V}}{c_1} + E_4\frac{\overline{u_1}}{c_1} + E_5\frac{\overline{p_1}}{\gamma p_1} + E_6\frac{\overline{B_1}}{B_1} \quad (C.4)$$

with a new definition of coefficients; this gives

$$\frac{\overline{\rho_2}}{\rho_2} = E_3\frac{\overline{V}}{c_1} + E_4\frac{\overline{u_1}}{c_1} + E_6\frac{\overline{B_1}}{B_1} + E_7\frac{\overline{p_1}}{\gamma p_1} \quad (C.5)$$

where $E_7 = 1 + E_5$.

Since $\tau = \dfrac{\theta\,\sigma-1-\gamma m_1^{\,2}(1-\sigma)^3/2}{\theta-\sigma}$,

$$\frac{\bar{\tau}}{\tau} = \{\frac{\sigma(\theta^2-1) + \gamma m_1^{\,2}\sigma\,[3\theta-1-6\theta\sigma + 3(\theta+1)\sigma^2 - 2\sigma^3]/2}{\theta\sigma-1-\gamma m_1^{\,2}(1-\sigma)^3/2}\}\frac{\bar{\sigma}}{\sigma} - [\frac{\gamma m_1^{\,2}(1-\sigma)^3}{\theta\sigma-1-\gamma m_1^{\,2}(1-\sigma)^3/2}]\frac{\bar{m}_1}{m_1} \qquad (C.6)$$

Substitution of (C.2) and (C.4) into (C.6) gives

$$\frac{\bar{p}_2}{\gamma p_2} = \frac{\bar{p}_1}{\gamma p_1} + \{\frac{\sigma(\theta^2-1)+\gamma m_1^{\,2}\sigma[3\theta-1-6\theta\sigma+3(\theta+1)\sigma^2-2\sigma^3]/2}{\theta\sigma-1-\gamma m_1^{\,2}(1-\sigma)^3/2}\}.$$

$$[E_3\frac{\bar{V}}{c_1} + E_4\frac{\bar{u}_1}{c_1} + E_5\frac{\bar{p}_1}{\gamma p_1} + E_6\frac{\bar{B}_1}{B_1}] - [\frac{\gamma m_1^{\,2}(1-\sigma)^3}{\theta\sigma-1-\gamma m_1^{\,2}(1-\sigma)^3/2}][\frac{\bar{B}_1}{B_1} - \frac{\gamma}{2}\frac{\bar{p}_1}{\gamma p_1}] \equiv$$

$$E_8\frac{\bar{V}}{c_1} + E_9\frac{\bar{u}_1}{c_1} + E_{10}\frac{\bar{B}_1}{B_1} + E_{11}\frac{\bar{p}_1}{\gamma p_1} \qquad (C.7)$$

Since $B_2/B_1 = \sigma$,

$$\frac{\bar{B}_2}{B_2} = E_3\frac{\bar{V}}{c_1} + E_4\frac{\bar{u}_1}{c_1} + E_5\frac{\bar{p}_1}{\gamma p_1} + (1+E_6)\frac{\bar{B}_1}{B_1} \qquad (C.8)$$

Since $V\text{-}u_1 = \sigma\,(V\text{-}u_2)$,

$$\frac{\bar{u}_2}{c_2} = [\frac{c_1}{c_2} - \frac{M_2}{M_1} + M_2 E_3]\frac{\bar{V}}{c_1} + [\frac{M_2}{M_1} + M_2 E_4]\frac{\bar{u}_1}{c_1} + M_2 E_5\frac{\bar{p}_1}{\gamma p_1} + M_2 E_6\frac{\bar{B}_1}{B_1} \equiv$$

$$E_{12}\frac{\bar{V}}{c_1} + E_{13}\frac{\bar{u}_1}{c_1} + E_{14}\frac{\bar{B}_1}{B_1} + E_{15}\frac{\bar{p}_1}{\gamma p_1} \qquad (C.9)$$

with a new definition of constants and noting that $M_2^2 = M_1^2/\sigma\ \tau$ and $c_2^2 = \tau\ c_1^2/\sigma$. Equations (C.5), (C.7) (C.8), (C.9)give the required perturbations.

Finally, if the shock path is written as $x = Vt + \psi(y,t)$, where ψ is the small perturbation and y the direction parallel to the shock, continuity of q, the velocity component parallel to the shock, gives.

$$\frac{\bar{q}_2}{V} = \frac{\bar{q}_1}{V} - \frac{\bar{u}_1}{V}\,\psi_y \qquad (C.10)$$