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ON A CLASS OF QUASIEQUILIBRATED FINITE DYNAMIC DEFORMATIONS OF SOLID CIRCULAR CYLINDERS

M. SHAHINPOOR,

Department of Mechanical Engineering, Pahlavi University, Shiraz, Iran

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Abstract. Governing equations for finite dynamic response of a special class of quasi-equilibrated motion of solid rubber circular cylinders are obtained.

An explicit relation between the radial time dependent displacement and the axial time dependent stretch is obtained. It is shown that the distribution of radial and axial stresses are parabolic with respect to the radial and axial material coordinates, respectively. An analysis of the motion in the phase-plane is given to prove that the motion is periodic. Exact solution for the period of oscillations is obtained in case the oscillations are free, but initiated with a large initial information, and when the axial loading is of the Heaviside step loading type. It is also shown that due to the Poynting and Kelvin effects of finite shear additional surface tractions must be maintained on the surfaces of the cylinder in order to provide the class of nonhomogeneous quasiequlibrated large dynamic deformations under discussion.

Introduction

Considerable attention has been focused in the last decade on problems associated with large dynamic deformations of bounded media. Classes of materials for which such deformations are possible in the elastic range are hyperelastic incompressible materials and in particular rubbers and rubber-like materials. The behaviour of such materials are fundamentally and inherently nonlinear in contrast to classical hard solids.

The first dynamic problem in finite elasticity of bounded media found its rather explicit but limited solution in two remarkable papers of Knowles[1, 2] analyzing large-amplitude radial oscillations of a thick-walled circular cylinder. Upon the assumption of incompressibility of the material, the governing equations could be integrated to reduce to an autonomous motion of a system with a single degree of freedom. Subsequently Hengs and Solecki[3] and independently Knowles and Jakub[4], investigated finite radial oscillations of spherical bodies. C. C. Wang[5] treated a similar problem for a thin-walled spherical shell. Wesolowski[6] has analyzed the problem of combined radial-axial motion of a cylinder of infinitesimal length apparently neglecting the secondary Poynting and Kelvin effects of finite shear. Finite longitudinal shear (telescopic) oscillations of thick-walled tubes were first studied by Nowinski and Schultz,[7] and Nowinski[8]. A similar problem was recently reconsidesed by Wang[9]. Some exact solutions to finite dynamic deformation in bounded hyperelastic media have been given by Shahinpoor and Nowinski[10], (1971) and Shahinpoor.[11, 13] In all these works except[9] for ref. 9 the motions represent a special class of dynamically possible deformation called quasiequilibrated motions of perfectly elastic incompressible bodies. Generalized universal solutions for quasiequilibrated motions were first given by Truesdell[14] and later revised in 1963 and 1968.

Truesdell and Noll[15] aiso give a brief review of the above-mentioned theory. In short these motions represent a special class of dynamically possible deformations for which the deformation field is circulation preserving and at each instant of time, the instantaneous configuration of the body is also a possible static configuration controllable by surface tractions only. The usual procedure for approaching these controllable deformations has been an inverse procedure, viz. the dynamic deformations are specified at the outset to be circulation preserving according to some permissible functions and no constraints on the boundaries may be assumed in advance. Suitable surface tractions must be maintained on the boundaries to provide such nonhomogeneous dynamic deformations. These boundary conditions in a sense compare with St. Venant's type boundary condititions in classical elasticity. Wang[9] mentions that from a practical point of view, such surface tractions are rather difficult to be realized mechanically.

Guided by the foregoing remarks in the present work we analyze the class of dynamically circulation preserving finite deformations of a solid circular cylinder made of an isotropic, incompressible, and hyperelastic material. The deformations are such that the a circular cylinder remains a circular cylinder at all times but undergoes large dynamic axial and radial deformations. Governing equations of finite dynamic response of the solid cylinder are obtained in a fixed spatial coordinate system. The fixed spatial coordinate formalism of finite dynamic elasticity was chosen in order to avoid a possible confusion which arose in Knowle's work[1, 2] and rectified later by Tadjbakhsh and Toupin[16], as regards the noninertial convicted coordinate formalism. For the corresponding equilibrium problem the reader is referred to refs. 17-19.

An explicit formula relating the radial time dependent displacement and the axial time dependent stretch is obtained. It is shown that the distribution of radial and axial stresses are parabolic with respect to the radial and axial material coordinates respectively. Analysis of the motion in the phase-plane is given to show that periodic motions of the above nature are possible. Exact expressions for the period of oscillations are obtained when the oscillations are free, but initiated with a large initial deformation and when the axial loading is of the Heaviside step loading type. The desired boundary tractions to provide such nonhomogeneous deformations are found to have a degree of arbitrariness to the extent of uncoupled portions of time and respective coordinates.

General Formulae

Let X^α, x^i denote the material and spatial coordinates of a body, respectively such that the motion of a body B_0 at $t=0$ to B at time $t > 0$ is represented by three real valued functions

$$x^i = x^i \ (X^\alpha, t), \ i, \alpha = 1, 2, 3 \qquad \ldots(1)$$

The equations of motion can be put into Cauchy's form:

$$t^{ij}\ ; j = \rho \ddot{x}^i, \ i, j = 1, 2, 3 \qquad \ldots(2)$$

or, alternatively

$$t^{ij}, j + \Gamma^k_{kj} \ t^{ij} + \Gamma^i_{jk} \ t^{jk} = \rho \left[\frac{\partial^2 x^i}{\partial t2} + \Gamma^i_{jk} \dot{x}^j \dot{x}^k \right], \qquad \ldots(3)$$

where t^{ij} are the components of the Cauchy stress tensor and Γ^i_{jk}, s are Christoffel symbols of the second kind, based on g_{ij}, that is, the metric tensor of x^i – system, and ρ is the density of the material. Stress tensor t^{ij} is constructed from t^i_j as

$$t^{ij} = g^{jk} \ t^i_k, \ i, j, k = 1, 2, 3 \qquad \ldots(4)$$

If the material is isotropic, homogenous and elastic, it possesses a strain energy function

$$\Sigma = \Sigma \ (I_{\bar{c}1}, II_{\bar{c}1}, III_{\bar{c}1}), \qquad \ldots(5)$$

so that by Green's energy method,[17]

$$t^k_l = b_{\bar{1}} \ c^{1k}_1 + b_o \ \delta^k_l + b_l \ c^k_l \qquad \ldots(6)$$

where

$$b_{\bar{1}} = \frac{2}{\sqrt{III_{\bar{c}1}}} \ \frac{\partial \Sigma}{\partial I_{\bar{c}1}}, \qquad \ldots(7)$$

$$b_{-o} = \frac{2}{\sqrt{III_{\bar{c}1}}} \left[II_{\bar{c}1} \frac{\partial \Sigma}{\partial II_{\bar{c}1}} + III_{\bar{c}1} \frac{\partial \Sigma}{\partial III_{\bar{c}1}} \right] \qquad \ldots(8)$$

$$b_{\bar{1}} = -2\sqrt{III_{\bar{c}1}} \ \frac{\partial \Sigma}{\partial III_{\bar{c}1}}. \qquad \ldots(9)$$

In the equations above $\bar{c}^{1i}_j$, s are the components of Couchy's strain tensor given by

$$\bar{c}^{1i}_k = g_{jk} \ \bar{c}^{1ij} = g_{jk} \ G^{\alpha\beta} \frac{\partial x^i}{\partial X^\alpha} \ \frac{\partial x^j}{\partial X^\beta} \qquad \ldots(10)$$

and $I_{\bar{c}1}$, $II_{\bar{c}1}$, $III_{\bar{c}1}$ are the invariants of $\bar{c}^{1i}_j$, given by

$$I_{\bar{c}1} \equiv \bar{c}^{1i}_i \ ; \ II_{\bar{c}1} = \tfrac{1}{2} \left[I^2_{\bar{c}1} - \bar{c}^{1k}_m \ \bar{c}^{1m}_k \right] \qquad \ldots(11)$$

$$III_{\bar{c}1} \equiv \det. \ \bar{c}^{1i}_j = | \bar{c}^{1k}_j | . \qquad \ldots(12)$$

Clearly

$$c^i_j = [\bar{c}^{1i}_j]^{-1} = II_{\bar{c}1} \ \delta^i_j - I_{\bar{c}1} \ \bar{c}^{1i}_j + \bar{c}^{1i}_k \ \bar{c}^{1k}_j \qquad \ldots(13)$$

or alternatively

$$c^i_j = \tfrac{1}{2} \ \epsilon^i_{rs} \ \epsilon^{pq}_j \ \bar{c}^{1r}_p \ \bar{c}^{1s}_q. \qquad \ldots(14)$$

where ϵ^i_{rs} and ϵ^{kq}_j are permutation symbols. In the case of incompressibility of the material

$$III_{\bar{c}1} = | \bar{c}^{1i}_j | = | c^i_j | = 1 \qquad \ldots(15)$$

and equation (6) reduces to

$$t^i_j = -p\delta^i_j + 2\frac{\partial \Sigma}{\partial I_{\bar{c}1}} \ \bar{c}^{1i}_j - 2 \frac{\partial \Sigma}{\partial II_{\bar{c}1}} \ c^i_j \qquad \ldots(16)$$

where p is of the nature of hydrostatic pressure and since it involves the term $\frac{\partial \Sigma}{\partial III_{\bar{c}1}} \ | \ III_{\bar{c}1} = 1$, it should be determined later from equations of motion and boundary conditions.

Formulation of the Problem

Let us consider a solid circulur cylinder made of homogeneous, isotropic, ideally elastic and incompressible material. Let the cylinder points have material coordinates $X^{\alpha}=R, H, Z$, and the cylinder and outer radius R_0 and height H in its undeformed state.

It is assumed that after the deformation the cylinder retains the shape of a circular cylinder and a point of the cylinder at time $t > 0$ has spatial coordinates $x^i = r,\ \theta,\ z$, such that the dynamic motion is axisymmetric and described by the functions

$$r=\eta\ (R, t),\ \theta=\theta,\ z=\xi\ (Z, t). \qquad ..(17)$$

From equations (10), (11) and (12) we arrive at

$$\bar{c}^{-1}{}^i_j = \begin{bmatrix} \eta_R^2 & 0 & 0 \\ 0 & \frac{\eta^2}{R^2} & 0 \\ 0 & 0 & \xi_Z^2 \end{bmatrix} ; \quad c^i_j = \begin{bmatrix} \frac{\eta^2}{R^2}\xi_Z^2 & 0 & 0 \\ 0 & \frac{R^2}{\eta^2} & 0 \\ 0 & 0 & \frac{\eta^2\eta^2 R}{R^2} \end{bmatrix} \qquad ..(18)$$

$$I_{\bar{c}^1} = \eta_R^2 + \xi_Z^2 + \frac{\eta^2}{R^2} \qquad ..(19)$$

$$II_{\bar{c}^1} = \frac{\eta^2}{R^2}\,[\,\eta_R^2 + \xi_Z^2\,] + \eta_R^2\ \xi_Z^2$$

$$III_{\bar{c}^1} = \frac{\eta_R^2\,\xi_Z^2\,\eta^2}{R^2} = 1, \text{ (incompressiblity).}$$

The incompressibility equation (15) in the present case reduces to

$$\xi\ (Z, t) = \lambda\ (t)\ Z,$$

$$\eta^2 = \frac{R^2}{\lambda}. \qquad ...(20)$$

In view of equation (20) equation (19) takes the form

$$I_{\bar{c}^1} = \frac{2}{\lambda} + \lambda^2,$$

$$II_{\bar{c}^1} = 2\lambda + \frac{1}{\lambda^2}, \qquad ...(21)$$

and consequently from (16), and (17-21),

$$t^{11} = -p\ (R, Z, t) + 2\,\frac{\partial\Sigma}{\partial I_{\bar{c}^1}}\,\frac{1}{\lambda} - 2\frac{\partial\Sigma}{\partial II_{\bar{c}^1}}\,\lambda,$$

$$\eta^2 t^{22} = -p\ (R, Z, t) + 2\,\frac{\partial\Sigma}{\partial I_{\bar{c}^1}}\,\frac{1}{\lambda} - 2\frac{\partial\Sigma}{\delta\, II_{\bar{c}^1}}\,\lambda, \qquad ...(22)$$

$$t^{33} = -p(R, Z, t) + 2\,\frac{\partial\Sigma}{\partial I_{\bar{c}^1}}\lambda^2 - 2\frac{\partial\Sigma}{\partial II_{\bar{c}^1}}\,\frac{1}{\lambda^2},$$

$$t^{12} = t^{21} = t^{23} = t^{32} = t^{13} = t^{31} = 0. \qquad ...(23)$$

With the above stresses and remembering that the only nonvanishing Christoffel's symbols are $\Gamma^1_{22} = -\eta,\ \Gamma^2_{12} = \Gamma^2_{21} = 1/\eta'$, the equation of motion (3) simply reduce to

$$\frac{\partial t^{11}}{\partial R} = \rho\eta_R\ \ddot{\eta},$$

$$\frac{\partial t^{33}}{\partial Z} = \rho\ \xi_Z\ \ddot{\xi} \qquad ...(24)$$

Employing (20) we obtain the equations of motion (24) in the form

$$\frac{\partial t^{11}}{\partial R} = \rho\ R\,[\,\frac{3}{4}\,\frac{\dot{\lambda}^2}{\lambda^3} - \tfrac{1}{2}\,\frac{\ddot{\lambda}}{\lambda^2}\,], \qquad ...(25)$$

$$\frac{\partial t^{33}}{\partial Z} = \rho\ Z\,\lambda\,\ddot{\lambda}$$

The above equations can be integrated to give

$$t^{11} = \tfrac{1}{2}\,\rho\,R^2\,[\frac{3\dot{\lambda}^2}{4\lambda^3} - \tfrac{1}{2}\frac{\ddot{\lambda}}{\lambda^2}] + E_1\ (Z, t), \qquad ...(26)$$

$$t^{33} = \tfrac{1}{2}\,\rho\,Z^2\,\lambda\,\ddot{\lambda} + E_2\ (R, t), \qquad ...(27)$$

where $E_1\ (Z,\ t)$ and $E_2\ (R,\ t)$ arise as a result of integration.

Employing equation (16) and with the aid of (20) one arrives at

$$t^{11} = t^{33} - [\lambda - \frac{1}{\lambda^2}]\,[2\lambda\,\frac{\partial\Sigma}{\partial I_c} + 2\frac{\partial\Sigma}{\partial II_c}]. \qquad ...(28)$$

Note that since $I_{\bar{c}^1}$ and $II_{\bar{c}^1}$ are only time dependent (see equation 21), then $\frac{\partial\Sigma}{\partial I_c}$ and $\frac{\partial\Sigma}{\partial II_c}$ are functions of λ (t) or, generally, in this particular problem time dependent only.

A close inspection of equations (26), (27) and (28) shows that

$$E_1\ (Z, t) = \tfrac{1}{2}\,\rho\,\lambda\,\ddot{\lambda}\,Z^2 + g_1\ (t), \qquad ...(29)$$

$$E_2(R,t)=\tfrac{1}{2}\rho R^2\left[\frac{3\dot{\lambda}^2}{4\lambda^2}-\frac{\ddot{\lambda}}{2\lambda^2}\right]+g_2(t) \quad ...(30)$$

where $g_1(t)$ and $g_2(t)$ are arbitrary functions of time, depending on the prescribed surface tractions at the ends and on the curved surface of the cylinder. Moreover,

$$g_2(t)-g_1(t)=\left[\lambda-\frac{1}{\lambda^2}\right]\left[2\frac{\partial\Sigma}{\partial I_{c^{-1}}}+2\frac{\partial\Sigma}{\partial II_{c^{-1}}}\right]. \quad ...(31)$$

Upon employing $E_1(z,t)$, and $E_2(R,t)$ in (26) and (27) respectively, it is found that

$$t^{11}=\tfrac{1}{2}\rho R^2\left[\frac{3\dot{\lambda}^2}{4\lambda^3}-\frac{\ddot{\lambda}}{2\lambda^2}\right]+\tfrac{1}{2}\rho\lambda\ddot{\lambda}Z^2+g_1(t),$$

$$t^{33}=\tfrac{1}{2}\rho R^2\left[\frac{\dot{\lambda}^2}{4\lambda^3}-\frac{\ddot{\lambda}}{2\lambda^2}\right]+\tfrac{1}{2}\rho\lambda\ddot{\lambda}Z^2g_2(t), \quad ...(32)$$

The presence of the parabolic terms in Z and R in the above expressions is due to the Poynting and Kelvin effects. They clearly indicate that additional surface tractions must be applied on the surface of the cylinder in order to produce the desired large nonhomogenous dynamic deformations.

At this stage let us take

$$t^{33}\left(R,\pm\frac{H}{2},t\right)=A(t)+\tfrac{1}{2}\rho R^2\left[\frac{3\dot{\lambda}^2}{4\lambda^3}-\frac{\ddot{\lambda}}{2\lambda^2}\right], \quad ...(33)$$

where $A(t)$ is a given end pressure and the second term on the right hand side of (33) will be known upon the determination of $\lambda(t)$.

Now from (32) and (33) is found that

$$g_2(t)=A(t)-\frac{1}{8}\rho H^2\lambda\ddot{\lambda}, \quad ...(34)$$

and consequently by employing (34) in (30)

$$g_1(t)=A(t)-\tfrac{1}{8}\rho H^2\lambda\ddot{\lambda}\left[\lambda-\frac{1}{\lambda^2}\right]\left[2\frac{\partial\Sigma}{\partial I_{c^{-1}}}\lambda+2\frac{\partial\Sigma}{\partial II_{c^{-1}}}\right. \quad ...(35)$$

Suppose that outer curved surface of the cylinder is basically traction free but additional time dependent tractions parabolic in Z should be supplied to maintain the dynamic equilibrium. We thus take

$$t^{11}(R_0,Z,t)=\tfrac{1}{2}\rho\lambda\ddot{\lambda}Z^2. \quad ...(36)$$

From equations (31), (35) and (36) we arrive at the nonlinear-differential equation of motion in the form

$$\tfrac{1}{2}\rho R_0^2\left[\frac{3\dot{\lambda}^2}{4\lambda^3}-\frac{\ddot{\lambda}}{2\lambda^2}\right]+A(t)-\tfrac{1}{8}\rho H^2\lambda\ddot{\lambda}-\left[\lambda-\frac{1}{\lambda^2}\right].$$

$$\left[2\frac{\partial\Sigma}{\partial I_{c^{-1}}}\lambda+2\frac{\partial\Sigma}{\partial II_{c^{-1}}}\right]=0, \quad ...(37)$$

Solution of the above equation for λ would determine the stresses and the radial displacements. For clarity let us take the material of the cylinder to be of the Moony-Rivlin type whose strain energy function is in the form

$$\Sigma=\tfrac{1}{2}\alpha(I_{c^{-}}-3)+\tfrac{1}{2}\beta(II_{c^{-1}}-3). \quad ...(38)$$

Then equation (37) is rewritten in the form

$$\tfrac{1}{2}\rho R_0^2\left[\frac{3\dot{\lambda}^2}{4\lambda^4}-\frac{\ddot{\lambda}}{2\lambda^3}\right]+\frac{A(t)}{\lambda}-\tfrac{1}{8}\rho H^2\ddot{\lambda}$$

$$-\alpha\left(\lambda-\frac{1}{\lambda^2}\right)-\beta\left(1-\frac{1}{\lambda^3}\right)=0. \quad ...(39)$$

This nonlinear second order ordinary differential equation can be integrated to give:

$$-\tfrac{1}{8}\rho R_0^2\frac{\dot{\lambda}^2}{\lambda^3}+\int_0^t\frac{A(t)\dot{\lambda}}{\lambda}dt-\tfrac{1}{2}\rho H^2\dot{\lambda}^2$$

$$-\alpha\left(\tfrac{1}{2}\lambda^2+\frac{1}{\lambda}\right)-\beta\left[\lambda+\frac{1}{2\lambda^2}\right]=\text{const.} \quad ...(40)$$

To be more specific let us take $A(t)$ in the form of the Heaviside step as

$$A(t)=\begin{cases}A_0 & t>o\\ o & t<o\end{cases} \quad ...(41)$$

So that the tube is initially at rest but suddenly at time t= is confronted with a step of magnitude A_0 which is subsequently maintained (Fig. 1).

The initial conditions take the form

$$\lambda(o)=1, \qquad \dot{\lambda}(o)=0, \quad ...(42)$$

This simplifies the equation (40) to

$$\dot{\lambda}^2=\left[\frac{3/2(\alpha+\beta)\lambda^3+A_0\lambda^3\ln\lambda-\alpha[\tfrac{1}{2}\lambda^5+\lambda^2]-\beta[\lambda^4+\tfrac{1}{2}\lambda]}{\tfrac{1}{8}\rho R_0^2+\frac{1}{16}\rho H^2\lambda^3}\right] \quad ...(43)$$

which is the governing equation of motion of the cylinder subjected to an impulse load A_0 at time $t=0^+$.

Analysis of the Problem in the Phase Plane for Forced Oscillations

Equation (40) describes a trajectory C, Fig. 2, in the $\dot{\lambda}-\lambda$ phase plane associated with the motion. C is symmetric about the λ axis. According to the known theorems of oscillations $(T=\oint d\lambda/\dot{\lambda})$ is finite. The curve C starts at the initial point $\lambda_1=1$, $\dot{\lambda}=0$,

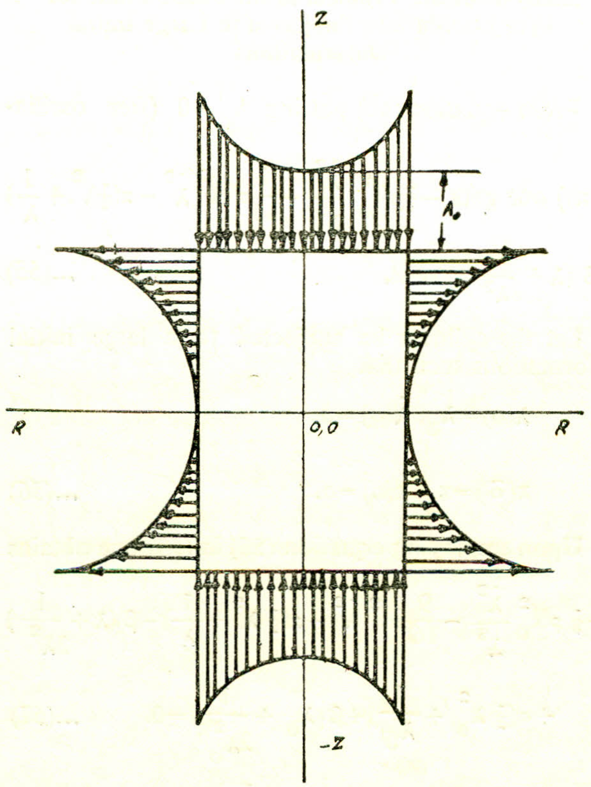

Fig. 1. Quasi-equilibrated configuration of the cylinder or a corresponding static equilibrium of the cylinder at a certain time t.

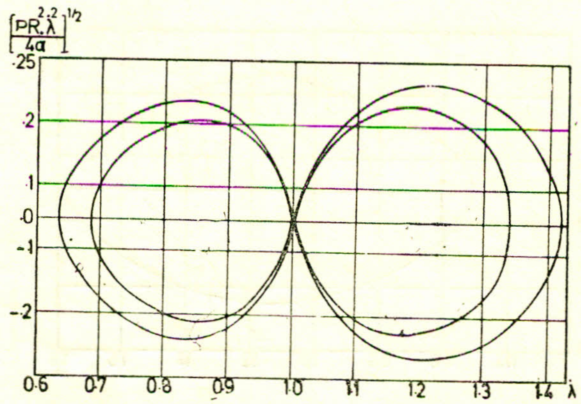

Fig. 2. Variation $\dot\lambda$ versus λ for $\beta=0$, $(H/R)=1$

at time $t=o$. If there exists another real positive root $\lambda_2 \neq \lambda_1$ such that $\dot\lambda = o$, then the integral curve becomes closed provided it is continuous and bounded in the interval $[\lambda_1, \lambda_2]$. Clearly, the curve C in equation (40) is continuous and bounded for all $0 < \lambda < \infty$. It may be shown that $\dot\lambda^2$ is monotonically increasing up to a finite maximum value and then monotonically increasing up to a finite maximum value and then monotonically decreasing to zero for which $\lambda = \lambda_2 \neq \lambda_1$. Therefore in the present case the existence of another positive root $\lambda_2 \neq \lambda_1$ is a necessary condition while boundedness of equation (40) is a sufficient condition for periodicity of motion. Putting $\dot\lambda = o$ in equation (40), yields

$$^3/_2\,(\alpha+\beta) - \alpha\left[\tfrac{1}{2}\lambda^2 + \frac{1}{\lambda}\right] - \beta\left[\lambda + \frac{1}{2\lambda^2}\right] + A_0 \ln\lambda = 0. \quad ..(44)$$

A closed form solution of the above equation for a positive root $\lambda_2 \neq \lambda_1 = 1$ cannot be obtained explicitly, but numerical solutions can easily be applied to find λ_2. In this regard let us rewrite equation (44) for a Neo-Hookean material $(\beta = 0)$ in the following form:

$$\left[\frac{\rho R_0^2 \dot\lambda^2}{4\alpha}\right] = \left\{\frac{\dfrac{^3/_2 + A_0}{\alpha}\ln\lambda - \tfrac{1}{2}\lambda - \dfrac{1}{\lambda}}{^1/_2\,\lambda^3 + \tfrac{1}{4}H^2/R_0^2}\right\} \quad ..(45)$$

Let us denote the r. h. s. of (45) by $f(\lambda, A_0/\alpha, H/R_0$, where

$$f(1, A_0/\alpha, H/R_0) = 0 \quad ..(46)$$

It may be shown that for a fixed H/R_0 and A_0/α the function f is monotonically increasing up to a maximum value and then monotonically decreasing to zero for which $\lambda = \lambda_2 \neq 1$. This is also shown schematically in Fig. 2, for a $H/R = 1$ and four different values of A_0/α for which the associated values of λ_2 are found. For small values of $(\lambda - 1)$ one may write, upon the expansion of equation (44), the following characteristic equation:

$$\lambda^2 + \lambda - 2\,(1 - A_0/\alpha)^{-1} = 0, \quad ..(46)$$

The only positive root $\lambda_2 \neq 1$ is found as

$$\lambda_2 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 8\,(1 - A_0/\alpha^{-1}} \quad ..(47)$$

It is clear that for λ_2 to be real

$$A_0/\alpha < 1, \quad ..(48)$$

which is a rather natural restriction on the amplitude and the sign of the applied loading for the prescribed oscillations to exist.

In fact if the value of A_0 is negative (compressive impact) then equation (48) is automatically satisfied and there are no restrictions on the magnitude of this compressive impact. However if $A_0 > 0$ (tensile impact), then the condition in equation (48) has to be satisfied ; otherwise the prescribed dynamic deformations cease to exist.

So far it was shown that the cylinder under a Heaviside step axial loading executes periodic oscillations in the form of combined axial-radial oscillations of the quasiequilibrated type, provided suitable surface tractions are maintained on the surfaces of the cylinder ; moreover the axial oscillations consist of a uniform dynamic stretch.

Having found λ_2 we can now express the period of oscillations simply in the form

$$T = \int \frac{d\lambda}{\dot{\lambda}} = 2\,\mathrm{sgn}\,(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\dot{\lambda}}\,, \quad \ldots(49)$$

and substituting for $\dot{\lambda}$ in equation (44) from equation (43) (simplified for the Neo-Hookean material) we get

$$T = 2\mathrm{sgn}\,(\lambda_2 - 1) \int_1^{\lambda_2} \left[\frac{.125\rho R_0^2 \lambda^{-3} + (1/16)\rho H^2}{1.5\alpha + A_0 \ln\lambda - \alpha(.5\lambda^2 + \lambda^{-1})} \right]^{\frac{1}{2}} d\lambda$$

...(50)

The convergence of this inproper integral of the second kind can be proved by the following p-tests for both limits :

for the limit $\lambda = 1$, $p = \frac{1}{2}$, $A_0 > o$

$$\lim_{\lambda \to 1^+} (\lambda - 1)^{\frac{1}{2}} \dot{\lambda}^{-1} = \left[\frac{\rho R_0^2 + \frac{1}{2}\rho H^2}{8A_0} \right]^{\frac{1}{2}}, \quad \ldots(51)$$

for the limit $\lambda = \lambda_2$, $p = \frac{1}{2}$, $A_0 > o$

$$\lim_{\lambda \to \lambda_2^-} (\lambda_2 - \lambda)^{\frac{1}{2}} \dot{\lambda}^{-1} = \left[\frac{\rho R_0^2 + \frac{1}{2}\rho H^2 \lambda_2^3}{-8A_0\lambda_2 + 8\alpha(\lambda_2^4 - 1)} \right]^{\frac{1}{2}}$$

for the limit $\lambda = 1$, $p = \frac{1}{2}$, $A_0 < o$

$$\lim_{\lambda \to 1^-} (1 - \lambda)^{\frac{1}{2}} \dot{\lambda}^{-1} = \left[\frac{-\rho R_0^2 - \frac{1}{2}\rho H^2}{8A_0} \right]^{\frac{1}{2}}, \quad \ldots(53)$$

for the limit $\lambda = \lambda_2$, $p = \frac{1}{2}$, $A_0 < 0$

$$\lim_{\lambda \to \lambda_2^+} (\lambda - \lambda_2)^{\frac{1}{2}} \dot{\lambda}^{-1} = \left[\frac{\rho R_0^2 + \frac{1}{2}\rho H^2 \lambda_2^3}{8A_0\lambda_2 + 8\alpha(\lambda_2^4 - 1)} \right]^{\frac{1}{2}} \quad \ldots(54)$$

Since all these limits are finite and $p < 1$ therefore the integral (50) converges in $[\lambda_1, \lambda_2]$.

Analysis of the Problem in the Phase Plane for Free Oscillations Subjected to Large Initial Deformations

From equation (40) putting $A_0 = 0$ (free oscillations) one gets

$$-\tfrac{1}{8}\rho R_0^2 \frac{\dot{\lambda}^2}{\lambda^3} - \frac{1}{16}\rho H^2 \dot{\lambda}^2 - \alpha(\tfrac{1}{2}\lambda^2 + \frac{1}{\lambda}) - \beta(\lambda + \frac{1}{2\lambda^2}) = \text{const.} \quad \ldots(55)$$

Let the cylinder be subjected to a large initial deformations such that

$$\lambda(o) = \lambda_0,\ \dot{\lambda}(o) = o,$$

$$x(\bar{o}) = x_0,\ \dot{x}(o) = o. \quad \ldots(56)$$

Upon employing equation (56) in (55) one obtains

$$-\tfrac{1}{8}\rho R_0^2 \frac{\dot{\lambda}^2}{\lambda^3} - \frac{9}{16} H^2 \dot{\lambda}^2 - \alpha(\tfrac{1}{2}\lambda^2 + \frac{1}{\lambda}) - \beta(\lambda + \frac{1}{2\lambda^2})$$

$$+ \alpha(\tfrac{1}{2}\lambda_0^2 + \frac{1}{\lambda_0}) + \beta(\lambda_0 + \frac{1}{2\lambda_0^2}) = 0. \quad \ldots(57)$$

Again the integral curve λ, $\dot{\lambda}$ (Fig. 3-4) represents a curve symmetric with respect to the λ-axis and intersecting the latter at the point $\lambda_1 = \lambda_0$, as

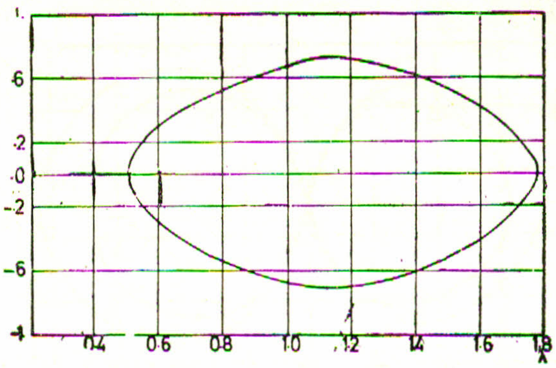

Fig. 3. Variation of $\dot{\lambda}$ versus λ for $\lambda_1 = 0.5$, $(H/R) = 1$.

expected. Following the argument leading to equation (44) we find that the motion under discussion is periodic if and only if there exists another real positive root $\lambda_2 \neq \lambda_0$ of the equation

$$-\alpha(\frac{1}{\lambda} + \tfrac{1}{2}\lambda^2) - \beta(\lambda + \frac{1}{2\lambda^2}) + (\alpha\frac{1}{\lambda_0} + \tfrac{1}{2}\lambda_0^2)$$

$$+ \beta(\lambda_0 + \frac{1}{2\lambda_0^2}) = 0, \quad \ldots(58)$$

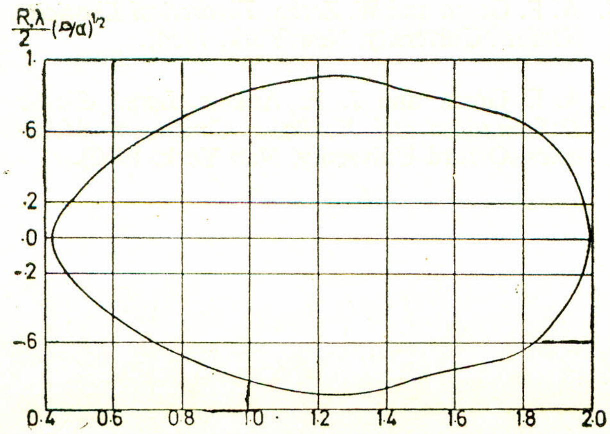

Fig. 4. Variation $\dot\lambda$ versus λ for $\lambda_1=2.$, (H/R)=1.

obtained from equation (57) by putting $\dot\lambda=0$. For a Neo-Hookean material the above equation simplifies to

$$\alpha(\lambda-\lambda_0)\,[1-\tfrac12\lambda_0^2\,\lambda-\tfrac12\lambda_0\,\lambda^2]=0, \qquad ...(59)$$

and the desired root λ_2 becomes

$$\lambda_2=\frac{\lambda_0}{2}\left[\sqrt{1+\frac{8}{\lambda_0^3}}-1\right] \qquad ...(60)$$

Consequently the period of oscillations is found in the form

$$T=\oint\frac{d\lambda}{\dot\lambda}-2\,\mathrm{Sgn}\,(\lambda_2-\lambda_1)\int_{\lambda_1}^{\lambda_2}\frac{d\lambda}{\dot\lambda} \qquad ...(61)$$

Substituting for $\dot\lambda$ in equation $\dot\lambda$ (61) from equation (57) (simplified to the Neo-Hookean material) we get

$$T=2\,\mathrm{sgn}\,(\lambda_2-\psi_0)\int_{\lambda_0}^{\lambda_2}\left[\frac{.125\,\rho_0^2\,\lambda^{-3}+0.063\,\rho H^2}{\alpha[.5\lambda_0^2+\lambda_0^{-1}-.5\lambda^2-\lambda^{-1}]}\right]^{\frac12}d\lambda \qquad ...(62)$$

as the period of oscillation of a cylinder subjected to large initial deformations. Again the convergence of this inproper integral of the second kind can be easily proved by the p-tests for integrals as follows :

for the limit $\lambda_0<1$, $p=\frac12$,

$$\lim_{\lambda\to\lambda_0^+}(\lambda-\lambda_0)^{\frac12}\,\dot\lambda^{-1}=\left[\frac{\rho R_0^2+(\frac12)\lambda_0^3\rho H^2}{8\alpha\,(1-\lambda_0^4)}\right]^{\frac12}, \qquad ...(63)$$

for the limit $\lambda_2>1$, $p=\frac12$,

$$\lim_{\lambda\to\lambda_2^-}(\lambda_2-\lambda)^{\frac12}\,\dot\lambda^{-1}=\left[\frac{\rho R_0^2+(\frac12)\lambda_2^3\rho H^2}{8\alpha\,(\lambda_2^4-1)}\right]^{\frac12} \qquad ...(64)$$

for the limit $\lambda_0>1$, $p=\frac12$,

$$\lim(\lambda_0-\lambda)^{\frac12}\,\dot\lambda^{-1}=\left[\frac{\rho R_0^2+(\frac12)\lambda_0^3\rho H^2}{8\alpha(\lambda_0^4-1)}\right]^{\frac12}, \qquad ...(65)$$

and for the limit $\lambda_2<1$, $p=\frac12$,

$$\lim(\lambda-\lambda_2)^{\frac12}\,\dot\lambda^{-1}=\left[\frac{\rho R_0^2+(\frac12)\lambda_2^3\rho H^2}{8\alpha\,(1-\lambda_2^4)}\right]^{\frac12}. \qquad ...(66)$$

since all these limits are finite and $p<1$, therefore the integral of equation (62) is convergent in $[\lambda_0,\lambda_2]$.

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