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VARIATIONAL CALCULUS AND STOKES' THEOREM

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Abstract. Simple calculus of variation has been used to prove Stokes' theorem. The method has the advantage of revealing physically the equality of surface integral of $\underline{\nabla} \times \underline{F}$ to a line integral of $\underline{F}$ around the boundary enclosing an area. The proposed method can also be extended to a general case when the dot product is replaced by a general multiplication.

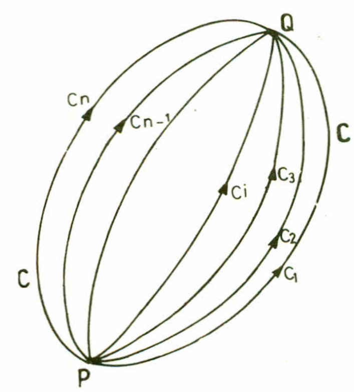
Stokes' theorem is a famous and useful theorem of vector analysis. It states that the line integral of the tangential component of a continuously differentiable vector point function $\underline{F}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\underline{F}$ taken over any surface S having C as its boundary

$$\oint \underline{F} \cdot d\underline{r} = \iint \underline{\nabla} \times \underline{F} \cdot d\_ \qquad (1)$$

A number of proofs using different methods are available in standard books[1—3] on the subject. Many of them use Green's theorem for a plane as a prerequisite to prove Stokes' theorem. A proof of the latter theorem is provided here which circumvents this difficulty. The proof is based on simple calculus of variation. It will be seen that the present method is elegant in that it shows more concretely, how the surface integral of $\underline{\nabla} \times \underline{F}$ over an area equals the line integral of $\underline{F}$ along the boundary enclosing the area. The procedure also shows that there exists a possibility of an extension of Stokes' theorem when the dot product in the line integral is replaced by either a cross product or a simple algebraic multiplication.

Proof of Stokes' Theorem

Let C be a simple closed curve bounding a surface S. Consider two fixed points P and Q on the curve. Devise a large number of paths C_1, C_2, $C_3 \ldots . C_i \ldots . C_n$ connecting the points P and Q such that C_1 and C_n together form the original boundary of the closed curve C. Thus



$$\oint \underline{F} \cdot d\underline{r} = \int_{C_i\, P}^{Q} \underline{F} \cdot d\underline{r} - \int_{C_n\, P}^{Q} \underline{F} \cdot d\underline{r} . \qquad (2)$$

Let us define

$$I_i = \int_{C_i\, P}^{Q} \underline{F} \cdot d\underline{r} , \qquad (3)$$

then

$$\oint \underline{F} \cdot d\underline{r} = I_1 - I_n$$

$$= (I_1 - I_2) + (I_2 - I_3) + \ldots (I_i - I_{i+1}) \ldots (I_{n-1} - I_n)$$

$$= \sum_{i=1}^{n-1} \delta I_i \qquad (4)$$

where

$$\delta I_i = I_i - I_{i+1} \qquad (5)$$

is the small variation in the values of the integrals between the two closely lying paths C_i and C_{i+1}. From Eq. (3)

$$\delta I_i = \delta \int_{C_i\, P}^{Q} \underline{F} \cdot d\underline{r} = \int_{C_i\, P}^{Q} \delta (\underline{F} \cdot d\underline{r}). \qquad (6)$$

where the integrand

$$\delta (\underline{F} \cdot d\underline{r}) = \delta \underline{F} \cdot d\underline{r} + \underline{F} \cdot \delta\, d\underline{r} . \qquad (7)$$

As far as the operation on the position vector is concerned, the d-operation and the δ-operation are interchangeable, so that

$$\delta (\underline{F} \cdot d\underline{r}) = \delta \underline{F} \cdot d\underline{r} + d (\underline{F} \cdot \delta\underline{r}) - d\underline{F} \cdot \delta\underline{r} \qquad (8)$$

Hence eq. (5) can be written as

$$\delta I_i = \int_{C_i\, P}^{Q} d (\underline{F} \cdot \delta\underline{r}) + \int_{C_i\, P}^{Q} (\delta \underline{F} \cdot d\underline{r} - d\underline{F} \cdot \delta \underline{r})$$

The first integral above equals

$$[\underline{F} \cdot \delta \underline{r}]_P^Q = O,$$

because there is no variation at the fixed points P and Q. Hence

$$\delta I_i = \int (\delta \underline{F} \cdot d\underline{r} - d\underline{F} \cdot \delta \underline{r}). \qquad (9)$$

Now

$$\delta \underline{F} = \frac{\partial \underline{F}}{\partial x} \delta x + \frac{\partial \underline{F}}{\partial y} \delta y + \frac{\partial \underline{F}}{\partial z} \delta z = (\delta \underline{r} \cdot \underline{\nabla}) \underline{F}.$$

Similarly

$$d \underline{F} = \frac{\partial \underline{F}}{\partial x} dx + \ldots\ldots = (d\underline{r} \cdot \underline{\nabla}) \underline{F}.$$

We have, therefore, from (9)

$$\delta I_i = \int_{C_i} \Big[(\delta \underline{r} \cdot \underline{\nabla}) \underline{F} \cdot d\underline{r} - (d\underline{r} \cdot \underline{\nabla}) \quad \cdot \delta \underline{r} \Big] \qquad (10)$$

Consider the expression

$$(\delta \underline{r} \times d\underline{r}) \cdot \underline{\nabla} \times F$$

and imagine $\underline{\nabla}$ to behave like a vector. Applying the vector identity

$$(\underline{A} \times \underline{B}) \cdot (\underline{C} \times \underline{D}) = (\underline{A} \cdot \underline{C})(\underline{D} \cdot \underline{B}) - (\underline{B} \cdot \underline{C})(\underline{D} \cdot \underline{A})$$

to this expression we obtain

$$(\delta \underline{r} \times d\underline{r}) \cdot \underline{\nabla} \times \underline{F} = (\delta \underline{r} \cdot \underline{\nabla}) \underline{F} \cdot d\underline{r} - (d\underline{r} \cdot \underline{\nabla}) \underline{F} \cdot \delta \underline{r} \cdot (11)$$

From Eqs. (10) and (11) we have

$$\delta I_i = \int_{C_i} (\underline{\nabla} \times \underline{F}) \cdot (\delta \underline{r} \times d\underline{r}). \qquad (12)$$

From Eq. (4) we, then, get

$$\oint \underline{F} \cdot d\underline{r} = \underset{n \to \infty}{\mathrm{Lt}} \sum_{i=1}^{n} \delta I_i$$

$$= \underset{n \to \infty}{\mathrm{Lt}} \sum_{i=1}^{n} \int_{C_i} (\underline{\nabla} \times \underline{F}) \cdot (\delta \underline{r} \times d\underline{r}).$$

Therefore

$$\oint \underline{F} \cdot d\underline{r} = \iint \underline{\nabla} \times \underline{F} \cdot d\underline{S}, \qquad (13)$$

which is the Stokes' theorem.

The method described above can also be generalised. Suppose the symbol $*$ stands for dot, cross or simple multiplication. Then, as in eq. (3), we could define

$$I_i = \int_{C_i\, P}^{Q} d\underline{r} * \underline{F}$$

where r and F would be scalars, $\underline{r}$ and $\underline{F}$ in case of a simple multiplication. The procedure adopted in the above method will still be valid up to Eq. (10), the integrand whereof will now read as

$$\Big[(\delta \underline{r} \cdot \underline{\nabla}) d\underline{r} - (d\underline{r} , \underline{\nabla}) \delta \underline{r} \Big] * \underline{F} \qquad (14)$$

Using the vector identity

$$(\underline{A} \times \underline{B}) \times \underline{C} = (\underline{A} \cdot \underline{C}) \underline{B} - (\underline{B} \cdot \underline{C}) \underline{A},$$

from (14), we get

$$\int \Big[(\delta \underline{r} \times d\underline{r}) \times \underline{\nabla} \Big] * \underline{F} = \iint (d\underline{S} \times \underline{\nabla}) * \underline{F}. \quad (15)$$

Therefore, we would have the generalised Stokes' theorem

$$\oint d\underline{r} * \underline{F} = \iint (d\underline{S} \times \nabla) * \underline{F} \qquad (16)$$

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