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DECAY OF THE TOTAL ENERGY IN MAGNETOHYDRODYNAMICS

G. AHMADI

Pahlavi University, Shiraz, Iran

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Abstract. The decay of the kinetic and magnetic energies in magnetohydrodynamics is analysed. It is observed that in the case of nonconducting walls and finite conductivity of the fluid the energy of the system decays faster than exponential.

The decay of the kinetic and magnetic energies in a magnetohydrodynamic flow has attracted considerable interests due to its astrophysical application. Weiss[1] obtained numerical results which illustrate the time-dependent behaviour of an initially uniform magnetic field in a highly conducting fluid in eddy motion. Cowling[2] has shown that it is impossible for certain class of fluid motion to prolong the decay times indefinitely. Jady[3] studied the normal mode of decay of a magnetic field in a nonuniformly rotating conducting fluid.

The present work deals with the decay of the kinetic and magnetic energies of an incompressible electrically conducting fluid in a fixed bounded nonconducing region. The method employed in the present analysis follows that of Serrin,[4] Rao,[5] Shahinpoor and Ahmadi,[6,7] It is observed that for the present case the rate of the decay is faster than exponential if the magnetic Reynolds number is finite.

Basic Equations. The equations governing the motion of an incompressible, electrically conducting fluid in dimensionless form read:

$$\frac{\partial v}{\partial t}+v\cdot\nabla v=-\nabla\left(P+\tfrac{1}{2}R_H H^2\right)+\frac{1}{R_e}\nabla^2 v+R_H H\cdot\nabla H \qquad (1)$$

$$\frac{\partial H}{\partial t}=\nabla\times(v\times H)+\frac{1}{R_m}\nabla^2 H \qquad (2)$$

$$\nabla\cdot v=0\ , \qquad \nabla\cdot H=0 \qquad (3)$$

where v is the velocity vector, H is the magnetic field strength vector, and P is the hydrostatic pressure. The dimensionless groups are:

$$R_e=\frac{\rho V\cdot L\cdot}{\mu_m} = \text{Reynolds number} \qquad (4)$$

$$R_m=\frac{\rho V_o L_o}{\mu_m} = \text{Magnetic Reynolds number} \qquad (5)$$

$$R_H=\frac{M^2}{R_o R_m}\ , \ M=\mu_m H_o L_o (\sigma/\mu)^{\frac{1}{2}}= \text{Hartmann number} \qquad (6)$$

μ, μ_m, ρ and σ are viscosity, magnetic viscosity, fluid density and conductivity, respectively. L_o and V_o are the maximum length and velocity in the domain.

Analysis. Consider the flow of an incompressible electrically conducting fluid in a domain V bounded by the regular nonconducting surface S on which the velocity vanishes. That imposes the following boundary conditions:

$$v=0 \quad \text{on S} \qquad (7)$$

$$H=0 \quad \text{on S} \qquad (8)$$

for the case of zero external magnetic field.

Furthermore let us assume that both velocity vector v and the magnetic field-intensity H possess continuous second order derivative in v. The total energy of the system is

$$T=T_1+T_2 \qquad (9)$$

where T_1 is the kinetic energy of the fluid,

$$T_1=\tfrac{1}{2}\int v^{2+} \qquad (10)$$

and T_2 denotes the energy of the magnetic field,

$$T_2=\frac{R_H}{2}\int H^2\ . \qquad (11)$$

Taking total time derivative from (9) we find

$$\frac{dT}{dt}=\frac{dT_1}{dt}+\frac{dT_2}{dt}=\int\left[V:\frac{\partial v}{\partial t}+R_H H\cdot\frac{\partial H}{\partial t}\right] \qquad (12)$$

Employing equations (1)-(3) in (12) and making use of the standard vector identity

$$\nabla\cdot(A\times B)=B\cdot\nabla\times A-A\cdot\nabla\times B \qquad (13)$$

we find

$$\frac{dT_1}{dt}=-\int v\cdot\nabla v\cdot v-\frac{1}{R_e}\int(\nabla\times v)^2+R_H\int H\cdot\nabla H\cdot v-\frac{1}{R_e}\int_S(\nabla\times v)\times v\cdot ds \qquad (14)$$

$$\frac{dT_2}{dt}=R_H\Big[\int H\cdot\nabla v\cdot H-\int v:\nabla H-\frac{1}{R_m}\int(\nabla\times H)^2-\frac{1}{R}\int_S(\nabla\times H)\times H\cdot ds\Big]\ . \qquad (15)$$

Employing the boundary conditions (7) and (8) together with equations (3) in the above yields.

The conventional volume infinitesimal is omitted constantly in the integrals which are extended over the volume V.

$$\frac{dT}{dt}=-\frac{1}{R_e}\int(\nabla\times v)^2-\frac{R_H}{R_m}\int(\nabla\times H)^2 \tag{16}$$

According to Serrin [4] if $\nabla\cdot A=0$ in v and $A=0$ on S then

$$\int(\nabla\times A)^2 \geqq 80\int A^2 \tag{17}$$

Equation (16) then becomes

$$\frac{dT}{dt}\leqq-\frac{80}{R_e}\int u^2-\frac{80\,R_H}{R_m}\int H^2 \tag{18}$$

Now from (18) it is found that

$$\frac{dT}{dt}\leqq -aT \tag{19}$$

where

$$a = \text{minimum}\left\{\frac{80}{R_e},\frac{80}{R_m}\right\} \tag{20}$$

It is then clear that

$$T(t)\leqq T(0)\,e^{-at} \tag{21}$$

In other words the rate of decay is at least exponential. It is observed that for $R_m < R_e$ the rate of decay is similar to that of a nonconducting viscous fluid but for $R_m > R_e$ the rate of decay is slower than that of a corresponding simple viscous fluid. For the limit of very large magnetic Reynolds number, i.e. infinite conductivity, the rate of decay becomes much smaller and it would not be exponential.

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