AN ALGEBRA OF NONLINEAR OPERATORS ON L² SPACES

Nasiruddin Ahmed

Electronics Division, Atomic Energy Centre, Dacca

(Received May 18, 1967)

An algebra of a class of nonlinear operators (more general than those of Hammerstein) acting within the L² space is developed. It is shown that the algebra is closed under the operations of "addition", "scaler multiplication and product by composition".

Introduction

A modified Hammerstein[1,2] operator is defined as

$$A_H x\,(t) = \sum_{s=0}^{N} \int_{I^s} \!\!\int K_s(t;\, \tau_1, \ldots \tau_s) g(\tau_1, x(\tau_1)\,).\; g(\tau_s, x(\tau_s))\; d\tau_1 \ldots d\tau_s,\; t \varepsilon I \tag{1.1}$$

and a modified Uryson operator [1,2] is defined as

$$Ax\,(t) = \sum_{s=0}^{N} \int_{I^s} \!\!\ldots\! \int U_s\,(t; \tau_1 \ldots \tau_s;\, x\,(\tau_1);\, x\,(\tau_s)\,)\; d\tau_1 \ldots d\tau_s\; t\varepsilon I \tag{1.2}$$

It is clear that these two classes of operators represent a direct extension of the more familiar Hammerstein and Uryson operatorsdefined as

$$A_H x(t) = \int_I K(t,\tau)\, g(\tau,\, x(\tau))\; d\tau,\; t\varepsilon I$$

and $A\,x\,(t) = \int_I U(t,\, \tau,\, x(\tau))\, d\tau,\ t\varepsilon I$, respectively.

In the development of the algebra we will only consider the modified Hammerstein operator acting within the L^2 space. For that it is essential to study the continuity and boundedness properties of these operators.

Continuity and Boundedness of A_H

To prove the continuity and boundedness of the operator A_H we make use of the following proposition.[3]

Proposition 2.1.—If the operator G defined by $Gx(\,t\,) = g(t,\, x\,(\,t\,)\,)$ maps every element $x \varepsilon L^2$ into an element in L^2 then G is bounded and continuous and the following inequality holds,

$$|\, g(t,\, u)\,| \leq |\, z\,(\,t\,)\,| + \beta\,|\, u\,| \tag{2.1}$$

for any $u \varepsilon R$, $z \varepsilon L^2$ and $\beta \varepsilon\ (0, \infty\,)$.

We now present the following proposition on the continuity and boundedness of A_H.

Proposition 2.2.—If $N \leq \infty$ and the operator G satisfy the conditions of proposition 2.1 and the set of kernels $K_s\, \varepsilon L^2\, (I^{s+1})$ for each s and are measurable in all the variables on I^{s+1} then A_H acts within L^2 and is therefore bounded and continuous. Proof:

Let us rewrite the operator A_H as

$$A_H x = \sum_{s=1}^{N} A_{H_s} x\,, \tag{2.2}$$

where

$$A_{H_s} x\,(t) = \int_{I^s} \!\!\ldots\! \int K_s\,(t,\, \tau_1 \ldots \tau_s)\, g(\tau_1 x(\tau_1)).\, g(\tau_s, x(\tau_s)).\; d\tau_1 \ldots d\tau_s\;\; t\varepsilon I\,. \tag{2.3}$$

By use of Holder's inequality[4] in 2.3 we have, for almost all, $t \varepsilon I$

$$|\, A_{H_s} x(t)\,| \leq \left(\int_{I^s} \!\!\ldots\! \int |\, K_s\,(t;\, \tau_1, \ldots \tau_s)\,|^2 d\tau_1 \ldots d\tau_s \right)^{\frac{1}{2}} \left(\int_I |\, g\,(\tau, x(\tau))\,|^2\, d\tau \right)^{s/2} \tag{2.4}$$

and since $K_s \,\varepsilon\, L^2\, (I^{s+1})$ and it is measurable in t on I we have

$$\|A_{H_s} x\| \leq \|K_s\|\;\; (\|Gx\|\,)^s \tag{2.5}$$

Since, by proposition 2.1, the operator G satisfies the inequality 2.1, and $x \,\varepsilon\, L^2$, it is easily proved that

$$\|Gx\| \leq \|z\| + \beta\, \|x\| \tag{2.6}$$

The proof of the last inequality follows from the fact that $\mu(\,I\,/\,I_0) = 0$, where μ is the Lebesgue measure on I and $I_0 = \{\, t\, \varepsilon\, I : |\, x(\,t\,)\,| < \infty\}$.

By the use of Minkowski's inequality[4] and the inequalities 2.5 and 2.6 in 2.2 for $N < \infty$, we have

$$\|A_H x\| \leq \sum_{s=1}^{N} \|K_s\|\;\, (\,\|\, z\,\| + \beta\, \|x\|)^s \tag{2.7}$$

Now at the University of Ottawa, Ottawa 2, Canada.

This proves that A_H acts within L^2 and is therefore bounded. In this case the domain of the operator A_H is the whole of L^2 space. But if $N = \infty$ then the operator A_H is defined only on a closed bounded subset $D(A_H) \subset L^2$, where

$$D(A_H) = \left\{ x \varepsilon L^2 : \|x\| \leq \frac{R - \|z\|}{\beta} \right\}$$

and $R = \dfrac{1}{\overline{\lim} \sqrt[s]{\|K_s\|}}$

clearly it is necessary that $\|z\| < R$.

For the continuity of the operator A_H we observe that

$$\|A_H x_1 - A_H x_2\| \leq \sum_{s=1} s\|K_s\| (\sup_{x \varepsilon S_r} \|Gx\|)^{s-1} \|Gx_1 - Gx_2\| \qquad (2.8)$$

where

$$S_r = \left\{ x \varepsilon L^2 : \|x\| \leq r = \frac{R - \|z\|}{\beta} \right\}$$

The series 2.8 exists if 2.7 is assumed to be uniformly convergent $\forall x \varepsilon S_r$. Thus the continuity of A_H on S_r follows from that of G on L^2. This completes the proof.

Remarks.—If G is assumed to satisfy a Lipschitz condition with a constant $a > 0$ for all $t \varepsilon I$, then it can be shown that A_H satisfies a Lipschitz condition with a factor $\alpha(r)$, where

$$\alpha(r) = a \sum_{s=1} s\|K_s\| (\|z\| + \beta r)^{s-1},$$

and

$$\|A_H x_1 - A_H x_2\| \leq \alpha(r) \|x_1 - x_2\| \; \forall \, x_1, x_2 \varepsilon S_r \subset L^2.$$

The Algebra

Let us denote by $B(H,G,F \ldots)$, a non-empty set of continuous and compact operators of the class A_H acting on suitable closed and bounded subsets of the space L^2. $H,G,F \ldots$ etc. are the elements of B. Those algebraic properties of the set B, which are most important for engineering work, are stated here without making any attempt at a rigorous proof.

It may be shown that B is an additive Abelian group. Let us define a map $\psi_1 : B \times B \rightarrow B$ by $\psi_1(H,G) = H+G$, i.e., ψ_1 assigns to any two elements of B a new element which also belongs to B. The closure property with respect to addition holds if, and only if, the domain of the resulting operator R is defined as the interesection of the domains of H and G; that is R is defined on a subset $D(R)$ of L^2 such that

$$D(R) \subseteq D(H) \cap D(G)$$

and $\quad Rx = (H+G)x = Hx + Gx \; \forall x \varepsilon D(R)$. It is the second equality that requires proof.

Additive Closure.—Let $R_1 = H+G$ with $H,G \varepsilon B$. Then $R_1 \varepsilon B$ provided $D(R_1) \subseteq D(H) \cap D(G)$. In case H and G both belong to $B_N \subset B$, where B_N is defined as the set of all operators of type A_H consisting of finite series (i.e. $N < \infty$) only, then $D(R_1)$ is the whole of L^2 space.

A_2 *Associativity.*—$R_2 = H + (G+F) = (H+G) + F$.

and $R_2 \varepsilon B$ provided $D(R_2) = D(H) \cap D(G) \cap D(F)$.

A_3. $\forall H \varepsilon B$, $-H \varepsilon B$, and $D(H) = D(-H)$,

because $\forall x \varepsilon D(H)$, $\|Hx\| = \|-Hx\|$.

A_4. It can be proved that there exists a unique null element $\phi \varepsilon B$ such that for all $x \varepsilon L^2$ $\phi x = \theta$, where θ is the null of the L^2 space. By A_1 and A_3, $H + (-H) \varepsilon B$, hence $\forall x \varepsilon D(H)$, $Hx + (-Hx) = Hx - Hx = \theta \varepsilon L^2$. Since $D(H) \subset L^2$, $\phi \varepsilon B$, and $H + \phi = \phi + H = H$, $\forall H \varepsilon B$.

This shows that B is an additive Abelian group.

We can define another operation on the set B. Let us denote by F the field of real or complex numbers and let us define by ψ_2 a map such that $\psi_2 : F \times B \rightarrow B$ by $\psi_2(a,H) = a.H$, $\forall H \varepsilon B$ and $\forall a \varepsilon F$. The set B is closed with respect to the operation ψ_2, because $\forall x \varepsilon D(H)$, $(a.H)x = a.Hx$ and $\|(a.H)x\| = |a| \; \|Hx\|$.

Therefore $a.H \varepsilon B$ $\forall H \varepsilon B$ and $\forall a \varepsilon F$ and $D(a.H) = (D(H)$. It is easily verified that with respect to this operation, the following relations hold true.

A_5. $a.(G+F) = a.G. + a.F$ $\forall G, F \varepsilon B$ and $\forall a \varepsilon F$ with $D(a.G + a.F) = D(G) \cap D(F)$

A_6. $(a+b).G = a. G + b.G.$ $\forall G \varepsilon B$ and $\forall a, b \varepsilon F$, with $D((a+b).G)) = D(G)$.

A_7. $a.(b.G) = (a.b).G$ $\forall G \varepsilon B$ and $\forall a,b \varepsilon F$ with $D((a.b).G) = D(G)$.

A_8. $1.H = H$, $\forall H \varepsilon B$ and $1 \varepsilon F$.

We can also define an equivalence relation "$\simeq$" on the set B such that

$H+G\simeq H+F \longrightarrow G\simeq F \quad \forall H, G, F \in B$ and that $D(G)=D(F)$ and $\forall x \in D(G)$, $Gx=Fx$. This equality must be understood in the sense of the norm of the space L^2.

For example, in the case of volterra-Frechet operators [1,5] two operators are equivalent if, and only if, each of the corresponding kernels are pairwise equivalent in the L^2 sense.

Let G and F both belong to B and let

$$Gx= \sum_{s=0} \int_{I^s}\cdots\int K_s(t;\tau_1,..\tau_s)\,x(\tau_1)\ldots x(\tau_s)\,d\tau_1\ldots d\tau_s$$

with $t \in I$ and $x \in D(G)$

and

$$Fx= \sum_{s=0} \int_{I^s}\cdots\int L_s(t;\tau_1,..\tau_s)\,x(\tau_1)\ldots x(\tau_s)\,d\tau_1\ldots d\tau_s$$

with $t \in I$ and $x \in D(F)$,

then for G to be equivalent to F it is necessary and sufficient that $D(G)=D(F)$ and that $K_s=L_s$ *a.e.* on I^{s+1} for each $s=0,1,2\ldots$

In the case of operators of type A_H, the equivalence relation is not so straightforward.

In case, $K_s=L_s$ *a.e.* on I^{s+1} for all $s=0,1,2\ldots$ then g must be equal to f for all $x \in L^2$ and for almost all $t \in I$. Conversely, if $g=f$, $\forall x \in L^2$ and almost all $t \in I$, then for the equivalence of the corresponding operators it is necessary and sufficient that $K_s=L_s$ *a.e.* on I^{s+1} for each s. However, it is important to note that an operator G could be equivalent to an operator F, both belonging to the class A_H, without actually any of the kernels of G being equivalent to any of the corresponding kernels of F. This is simply due to the presence of the operators g and f.

All the relations A_1—A_8 resulting from the two operations ψ_1 and ψ_2 defined on B are precisely the postulates of a linear vector space in which an equivalence relation "$\simeq$" is also defined. Thus $B(H,G,F,\ldots \psi_1, \psi_2, \simeq, \phi)$ is a linear vector space whose elements are the set of all continuous bounded nonlinear operators defined on suitable subsets of the space L^2.

Another important operation that can be defined on the set B is "the product by composition".

Let H and $G \in B$, and let $\psi_3 : B\times B \rightarrow B$ be defined as

$$\psi_3(H,G)=HOG=R_3$$

so that

$$R_3x=(HOG)x=H(Gx), \quad \forall x \in D(R_3)$$

where $D(R_3) \subseteq D(G) \cap \{x, x \in D(G) : Gx \in D(H)\}$.

Thus, B is closed with respect to this operation provided $D(R_3)$ is chosen as defined above. We note the following properties of the set B with respect to the operation ψ_3:

A_9 : $\forall H,G \in B, \quad HOG \in B$

with $D(HOG)=D(G) \cap \{x, x \in D(G) : Gx \in D(H)\}$

A_{10} $\forall H,G,F \in B$

$$HO(GOF)=(HOG)\,OF=HOGOF$$

with $D(HOGOF)=D(F) \cap \{x, x \in D(F) : Fx \in D(G)\} \cap \{x, x\ D(F) : (GOF)x \in D(H)\}$

In general, the set B does not satisfy the left distributivity property but it does satisfy the right distributivity.

A_{11} $HO(G+F) \neq HOG+HOF$

The equality holds only when H is liCear.

A_{12} $(G+F)OH=GOH+FOH$
with $D((G+F)OH)=D(GOH) \cap D(FOH)$
$=D(H) \cap \{x, x \in D(H) : Hx \in D(G)\}$
$\cap \{x, x \in D(H) : Hx \in D(F)\}$. With res ect to the operation ψ_3, B is a semigroup and it would be a ring if A_{11} were true. We note that the set of all bounded linear operators defined on any Banach space forms a ring, which makes the study of linear operators comparatively simpler.

With respect to the operation ψ_3, we may define an identy element I by,

$$HOI=IOH=H$$

so that

$(HOI)x=(IOH)x=Hx \ \forall x \in D(H)$
with this new element included, B is an algebra closed under the operations of addition, multiplication by scalers, and product by composition. This algebra may now be denoted by $B(H,F,G,\ldots, \psi_1, \psi_2, \psi_3, \simeq, \phi, I)$, ψ_1, ψ_2, ψ_3 are the operations as defined "$\simeq$" is a relation, and ϕ and I are the two special elements of B. In this algebra

TABLE OF $D(HOG)$.

G	$D(G)=D(H)$	$D(G)\supset D(H)$	$D(G)\subset D(H)$	$D(G)\cap D(H)=\phi$
Reducing $GD(G)\subset D(G)$	$D(HOG)=D(H)$	$D(G)\supset$ $D(HOG)\supseteq D(H)$	$D(HOG)=D(G)$	—
Expanding $GD(G)\supset D(G)$	$D(HOG)\subset D(H)$	$D(HOG)\subset D(H)$	$D(HOG)\subseteq D(G)$	—
$GD(G)=D(G)$	$D(HOG)=D(H)$	$D(HOG)=D(H)$	$D((HOG)=D(G)$	—

the cancellation law does not hold since $HOG \simeq HOF$ does not imply that $G \simeq F$. This is true even in the case of linear operators. Therefore, the element I may not be unique. Also there may not exist inverses for the element of B since $y=Hx$, may not have a solution for x, for an arbitrary $y \in L^2$.

An important point associated with the operation ψ_3, defined on B, is the domain of the combined operator HOG for all H and G in B. The entries in the above table indicate the domain of the operator HOG. It will be clear that $D(HOG)$ is a function of the domains of the individual operators and the nature of the leading operator G.

The situation represented by the last column of the table may arise in case the zeroth order terms in our operators are present and are not identical.

Conclusion

In this paper the author has made an attempt to study the algebraic structure of an important class of operators. Some important fundamental properties of the totality B of all these operators have been presented. Special emphasis has been placed on the correct specification of the domain of any combination of elements of the set B. The need of this emphasis is felt immediately in the case of strong nonlinearities. An immediate application of this algebra can be found in the systems engineering.

It must be noted that the algebra developed in this paper assumes the ordinary continuity in the sense that each of the elements of the set B transforms every strongly convergent sequence into a strongly convergent one. This, however, is a strong restriction. The use of such concepts as weak continuity, weak lower and weak upper semicontinuities may lead to a weaker algebra.

The assumption of any of the Lp spaces as the domains of the operators also places a restriction on the strength of nonlinearity that can be handled. Strong nonlinearities can be handled, if the spaces on which the operators act is taken to be a suitable Orlicz space, which is a generalization of the L^p spaces.

Acknowledgement.—The author wishes to thank Professor G.S. Glinski of Ottawa University, Canada, for many helpful suggestions and comments on the material appearing here. The author would also like to thank Dr. I.H. Usmani, Chairman, Pakistan Atomic Energy Commission, for supporting these investigations.

References

1. N.U. Ahmed, *Functional Approach to Analysis and Synthesis of Nonlinear Systems*, Doctorate Thesis, Department of Electrical Engineering, Ottawa University, Ottawa, Canada, 1965. pp. 14–18.
2. G.S. Glinski and N.U. Ahmed, *Solution of a Class of Nonlinear Functional Equations Arising in Time Varying Nonlinear Feedback Control Systems*, T.R. 65–15, Department of Electrical Engineering, Ottawa University, Ottawa, Canada, 1965.
3. M.A. Krasnoselskii, *Topological Methods in Nonlinear Integral Equations* (Macmillan Co., New York, 1964) pp. 22–30.
4. P.R. Halmos, *Measure Theory* (D. Van Nostrand Co., New York, 1964), pp. 174–177.
5. Vito Volterra, *Theory of Functionals and Integral and Integrodifferential Equations* (Dover Publications, 1959), pp. 25.
6. G.J. Minty, *Monotone (Nonlinear) Operators in Hilbert Space*, Duke Mathematical Journal, **28-29**, 341–46 (1961-62).