NOTE ON A FULLY DYNAMICAL VARIANT OF LEES' DISC METHOD FOR THE RAPID DETERMINATION OF THE THERMAL CONDUCTIVITY OF POOR CONDUCTORS

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As an extension of the previously reported semi-dynamical method utilizing the linearity of the plot of $\partial\theta/\partial t$ against θ, the temperature of the standard brass disc, a fully dynamical variant has been developed by extrapolating this linear plot backwards to the initial condition at $t = 0$. The experimental results indicate the necessity for a significant correction for loss of heat from the periphery of the sample disc, an expression for which is derived. The experimental values obtaind from the fully dynamical measurements are shown to be in satisfactory agreement (within $\pm$ 4%) with that obtained by the equilibrium method. The resulting formula contains a small term involving the specific heat of the sample disc, which can thus be determined from two measurements, with a thin disc and a thick one.

1. Basis of the Fully Dynamical Method

In a previous communication,[1] the approach to equilibrium in the measurement of thermal conductivities by Lees' disc method was analyzed mathematically, and for time $t > 0.1\ d/E'_b$, where d is the thickness of the experimental disc and E'_b the effective emissivity of the brass disc, the following equation was derived

$$(m'c' + \tfrac{1}{3} mc)\frac{\partial\theta_d}{\partial t} = \frac{KA}{d}(\theta_s - \theta_d) - A'E'(\theta_d - \theta_{amb}), \quad (1)$$

where m is the mass of the experimental disc, A is its area of cross section and c its thermal capacity, θ_s is the temperature of the steam chest and θ_d that of the brass disc, and m', c', A', E', all refer to the brass disc (with $A'E' = A \times E'_b$). Experiments carried out on composite discs of different thicknesses (Fig. 1), give straight line plots for $\partial\theta_d/\partial t$ against θ_d in accordance with equation (1).

Another useful result is obtained if we extrapolate the linear portion of the graph in the other direction to $\theta_d = \theta_{amb}$ i.e. towards $t = 0$. Equation (1) then gives

$$(m'c' + \tfrac{1}{3} mc)\ (\partial\theta_d/\partial t)_{t\to o} = \frac{KA}{d}(\theta_s - \theta_{amb}), \quad (2)$$

and, if we also include the correction term given by the next approximation worked out in the previous paper, this becomes

$$\left\{ m'c' + \frac{1}{3}mc + \frac{1}{45}\frac{(mc)^2}{m'c'} \right\} (\partial\theta_d/\partial t)_{t\to o} = \frac{KA}{d}\ (\theta_s - \theta_{amb}),$$

$$\text{whence } K = \frac{d}{A(\theta_s - \theta_{amb})} \left\{ m'c' + \tfrac{1}{3}mc + \frac{1}{45}\frac{(mc)^2}{m'c'} \right\} (\partial\theta_d/\partial t)_{t\to o}$$

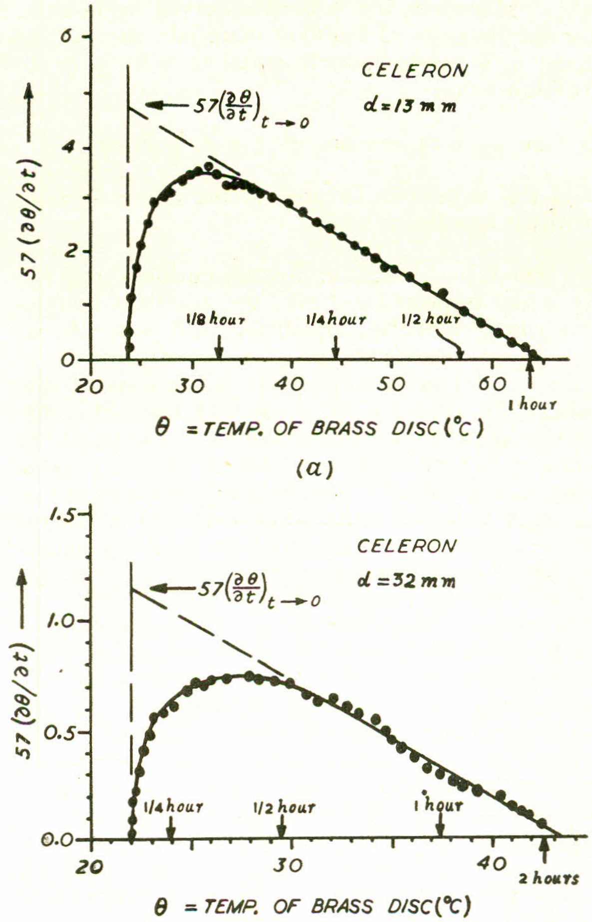

Fig. 1.—Experimental curves for $\partial\theta/\partial t$ against θ, the temperature of the brass disc, obtained with two thickness of "Celeron" discs, namely 13 mm. ($\frac{1}{2}''$) and 32 mm. ($1\frac{1}{4}''$). The horizontal arrow indicates the value of 57 ($\partial\theta/\partial t$) extrapolated linearly to the initial condition ($t = 0$).

$$= \left\{ 1 + \frac{1}{3}\frac{mc}{m'c'} + \frac{1}{45}\left(\frac{mc}{m'c'}\right)^2 \right\} \frac{m'c'd}{A(\theta_s - \theta_{amb})}$$

$$\times (\partial\theta_d/\partial t)_{t\to 0} \qquad (3)$$

This shows that a measure of the thermal conductivity can be obtained without separately measuring the emissivity of the brass disc, as is normally necessary in Lees' disc method. Of course, the factor

$$\left\{ 1 + \frac{1}{3}(mc/m'c') + \frac{1}{45}(mc/m'c')^2 \right\}$$

is uncertain to the extent that the specific heat, c, of the sample is not known accurately, but for a comparatively thin sample ($d < 2$ cm.) this uncertainty will be small, perhaps of the order of 2%. Moreover, it has been observed by us that, for the majority of building materials, the specific heat, c, is approximately equal to 0.23 ± 0.06. It follows that

$$mc \backsimeq 0.23 \times \text{mass} \backsimeq \tfrac{1}{4}\rho \times \text{Volume},$$

and this value can be used in the above formula without significant error.

The fully dynamical method outlined above is in many respects similar to the transient method described by Beatty, Armstrong, and Schoenborn[2] for rapid measurement of the conductivity of plastics and similar materials in the form of thin discs. However, in their method the metal disc is not allowed to radiate heat to the atmosphere, thus giving an ultimate solution of a type that does not allow of a steady state measurement as is done with the apparatus used in the present experiments, and the time lost before a simple linear formula becomes applicable to the transient flow is several times greater ($t > \frac{1}{2}d^2/K$) than for the method suggested above. The accuracy of both methods appears to be of the same order. Gafner[3] has also described a transient method for measuring the thermal conductivity of rocks and building materials, but he uses a heat sink, whose temperature is maintained a fixed number of degrees below that of the source. He claims an accuracy of $\pm$ 7%. The method described in the present paper can be said to be intermediate between the other two, because the metal disc automatically attains a limiting temperature intermediate between that of the heat source and the ambient temperature. The accuracy of the measurement attainable by this method is seen in the next section to be of the order of $\pm$ 4%.

2. Some Experimental Results

In order to examine in detail the applicability of equation (3), we define a quantity "$K_{dynamic}$" as

$$K_{dynamic} = \frac{d\,m'\,c'}{A(\theta_s - \theta_{amb})}(\partial\theta_d/\partial t)_{t\to 0}$$

$$= K \Big/ \left\{ 1 + \frac{1}{3}\frac{mc}{m'c'} + \frac{1}{45}\left(\frac{mc}{m'c'}\right)^2 \right\} \qquad (4)$$

Then, if the value of K determined from the equilibrium temperature of the brass disc is called "$K_{equilib}$", we have $K = K_{equilib}$ for a thin disc, and get, from equation (4),

$$K_{equilib}/K_{dynamic} = 1 + \frac{1}{3}\frac{mc}{m'c'} + \frac{1}{45}\left(\frac{mc}{m'c'}\right)^2$$

$$\backsimeq 1 + d \times \text{Constant}, \qquad (5a)$$

because for a given material, m c is proportional to the sample thickness, d. If we take the reciprocal, we get

$$K_{dynamic}/K_{equilib} = 1 - \frac{1}{3}\frac{mc}{m'c'} + \frac{4}{45}\left(\frac{mc}{m'c'}\right)$$

$$+ \ldots \backsimeq = 1 - d \times \text{Constant}. \qquad (5b)$$

Table 1 gives values of both the above ratios determined experimentally for values of d ranging from 1/4 inch up to 7/4 inches for discs of "Celeron", a laminated insulating material, typical graphs of $\partial\theta/\partial t$ against θ for which have already been shown in Fig. 1.

A graph of either of the two above ratios when plotted against 'd' should be nearly linear (the approximation being expected to be better for the first ratio, $K_{dynamic}/K_{equilib}$), and should extrapolate to unity for $d = 0$. Figures 2(a) and 2(b) show the experimental data of Table 1 plotted according to equations (5a) and (5b) respectively. (The measurements with the 1/4 inch thickness of celeron are given one third the weight of the others, the corresponding value of K determined by the equilibrium method being abnormally low, probably because the temperature of the brass disc becomes high enough (80°C.) to make the use of Newton's law somewhat erroneous). Both graphs do extrapolate to unity within the limits of experimental error, and the standard deviation of the experimental points indicates an accuracy of about $\pm$ 4%, cf. the standard deviation of 0.034 for the values of $K_{dynamic}/K_{equilib}$ in Table 1.

3. Influence of Peripheral Heat Loss

An interesting feature of the graphs of Fig. 2

is that, contrary to expectation, linearity is nearly exact in the case of equation (5b), while there is a very marked curvature for large values of d when ($K_{equilib}/K_{dynamic}$) is plotted against d. This discrepancy is largely due to the effect of the heat lost from the peripheral surface of the specimen, together with a contribution from the errors remaining in the second approximation for the derivation of equation 2 (cf equation 3). The peripheral heat loss is the more important factor, because it alters the first order terms in equations (5). This heat loss occurs in the equilibrium measurement as well as in the dynamical measurement, but the value of the correction factor for this effect is different in the two cases. The correction factor for the measurement by the equilibrium method was derived in a previous paper[4] (equation (14)), and the formula is

$$K_{equilib}/K = \left[1 - \frac{E}{K}\frac{d}{R} \times \left\{ \frac{d}{2} + d\ (\theta_d - \theta_{amb})/(\theta_s - \theta_{amb}) \right\} \right] \qquad (6a)$$

$$= 1 - \frac{E}{K}\frac{d}{R}\left(\frac{d}{2} + K_{equilib}/E'_b\right),$$

which can be recast to give

$$K_{equllib}/K = \left(1 - \frac{E}{K}\frac{d^2}{2R}\right) \div \left(1 + \frac{E}{K}\frac{d}{R} \times \frac{K}{E'_b}\right), \qquad (6b)$$

where E is the emissivity of the sample and K is the actual value of the thermal conductivity. An estimate of the corresponding factor for the measurement by the fully dynamical method under discussion can be obtained by the following simple argument.

An examination of the analysis[1] leading upto equation (1) describing the linear part of the graphs for $\partial\theta_d/\partial t$ against θ_d (cf. Fig. 1) shows that this linear part corresponds to the region in which the temperature distribution across the thickness of the sample disc can also be treated as linear (with the inclusion of a small second-order term). Thus, in this region, the relationship of the peripheral loss to the conduction flow of heat will be similar to that under the equilibrium condition, provided that due account is taken of the different temperature of the the disc. Therefore, for

TABLE 1.—COMPARISON OF MEASUREMENT BY EQUILIBRIUM AND FULLY DYNAMICAL METHODS (EXPERIMENTAL RESULTS FROM TWO SETS OF EXPERIMENTS).

Thickness of disc. (in.)	$K_{equilib} \times 10^3$ (cal. cm.$^{-1}$ sec.$^{-1}$/°C.)		$K_{dynamic} \times 10^3$ (cal. cm.$^{-1}$ sec.$^{-1}$/°C.)		$\frac{K_{equilib}}{K_{dynamic}}$	$\frac{K_{dynamic}}{K_{equilib}}$
1/4	0.383 0.387	0.385 ±.002	0.350 0.330	0.340 ±.010	$1.13_2 \pm .03$	$0.88_4 \pm 0.027$
1/2	0.406 0.423	0.414 ±.008	0.374 0.386	0.380 ±.006	$1.09_0 \pm .03$	$0.91_7 \pm 0.023$
3/4	0.396 0.392	0.394 ±.002	0.326 0.332	0.329 ±.003	$1.19_7 \pm .01$	$0.83_5 \pm 0.009$
1	0.384 0.359	0.372 ±.012	0.276 0.268	0.272 ±.004	$1.36_7 \pm .05$	$0.73_2 \pm 0.026$
5/4	0.347 0.318	0.332 ±.014	0.242 0.254	0.248 ±.006	$1.33_8 \pm .06$	$0.74_7 \pm 0.036$
3/2	0.313 0.316	0.314 ±.002	0.182 0.207	0.194_5 ±.013	$1.61_5 \pm .10$	0.619 ± 0.041
7/4	0.290	0.290	0.160	0.160	1.81	0.55_2
	Std. Dev. =0.009		Std. Dev. = 0.008			Std. Dev. = 0.034

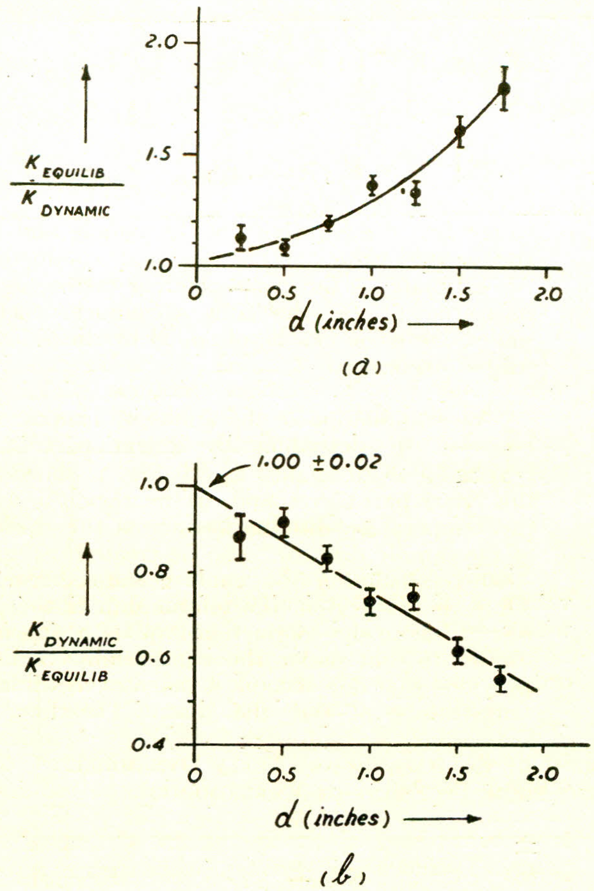

Fig. 2.—Plots of the ratios (a) $K_{equilib}/K_{dynamic}$ and (b) $K_{dynamic}/K_{equilib}$ as determined for different thicknesses of "Celeron", showing the nearly linear variation in the case of (b).

any intermediate position on the linear part of the $\partial\theta/\partial t$ against θ graph, say at a temperature θ'_d of the brass disc, the correction factor will be given by the expression (6a), but with θ'_d written in place of the equilibrium value θ_d. When the linear part of the graph is extrapolated (linearly) to $t = 0$ for the calculation of $K_{dynamical}$, then θ'_d becomes equal to θ_{amb}, the ambient temperature, and therefore from equation (6a) the correction factor for the peripheral loss simply reduces to

$$\left\{1 - \frac{E}{K}\frac{d}{R}\left(\frac{d}{2} + 0\right)\right\} = \left(1 - \frac{E}{K}\frac{d^2}{2R}\right). \tag{6c}$$

Introduction of this correcting factor into equation (4) gives us

$$K_{dynamic} = K\left(1 - \frac{E}{K}\frac{d^2}{2R}\right)$$

$$\div \left\{1 + \frac{1}{3}\frac{mc}{m'c'} + \frac{1}{45}\left(\frac{mc}{m'c'}\right)^2\right\} \tag{7}$$

$$= K_{equilib}\left(1 + \frac{E}{K}\frac{d}{R}\frac{K}{E'_b}\right)$$

$$\div \left\{1 + \frac{1}{3}\frac{mc}{m'c'} + \frac{1}{45}\left(\frac{mc}{m'c'}\right)^2\right\},$$

whence

$$K_{dynamic}/K_{equilib} = \left(1 + \frac{d}{R}\frac{E}{E'_b}\right)$$

$$\div \left\{1 + \frac{1}{3}\frac{mc}{m'c'} + \frac{1}{45}\left(\frac{mc}{m'c'}\right)^2\right\} \tag{8a}$$

$$\backsimeq 1 - \left(\frac{1}{3}\frac{mc}{m'c'} - \frac{d}{R}\times\frac{E}{E'_b}\right)$$

$$+ \frac{1}{3}\frac{mc}{m'c'}\left(\frac{4}{15}\frac{mc}{m'c'} - \frac{d}{R}\frac{E}{E'_b}\right). \tag{8b}$$

This equation shows that (i) the slope of the graph for ($K_{dynamic}/K_{equilib}$) against 'd' is considerably less than that given by the approximate equations (5), and (ii) the second-order term, proportional to d^2, also becomes small due to the mutual cancellation of the quantities within the brackets.

4. Indirect Estimation of the Thermal Capacity of Celeron

A particular advantage of the dynamical method is that by using the measurements on samples of two different thicknesses (cf. equation 7), or by combination with the equilibrium measurement (cf. equations 8), the thermal capacity of the material can be found. Let us compare the mean slope of the graph with the theoretical value from equation (8b), which gives, on putting $mc = \rho Ad \times c$,

$$-(\text{Mean Slope}) \simeq \left(\frac{1}{3}\frac{\rho Ac}{m'c'} - \frac{1}{R}\frac{E}{E'_b}\right)$$

$$\div \left(1 + \frac{1}{2}\frac{\rho Ac}{m'c'}\bar{d}\right),$$

whence

$$c\left(1 + \frac{3}{2}\bar{d} \times \text{Slope}\right)$$

$$= \frac{3m'c'}{\rho A}\left(\frac{1}{R}\frac{E}{E'_b} - \text{Slope}\right), \tag{9}$$

TABLE 2

Thickness of disc:	1/4″	1/2″	3/4″	1″	5/4″	3/2″	7/4″
$K \times 10^3$ from equation 7:	(0.390)	0.498	0.496	0.473	0.499	0.458	0.446
Mean of successive values		0.46_2	0.49_7	0.48_4	0.48_6	0.47_8	0.45_2

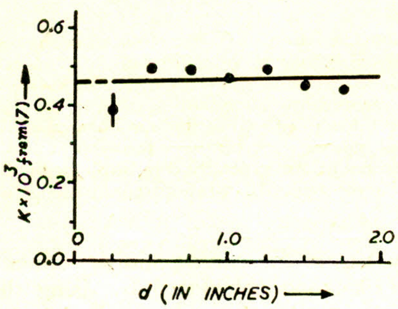

Fig. 3.—Graph showing the residual variation in the corrected value of 'K' as calculated from the fully dynamical measurements using equation (7).

Now the experimental value of the slope is

$$-\frac{0.48 \pm 0.06}{5} \text{ i.e.} - 0.096 \pm 0.012 \text{ cm.}^{-1}, \text{ while}$$

the other constants are[4]

$$\begin{aligned} m'c' &= 45.9 \text{ cal./g./°C.,} \\ A &= 76 \text{ sq. cm.} \\ R &= 5\text{cm.} \\ E/E'_b &= 0.053/0.29 = 0.18 \\ \rho &= \text{density of “Celeron”} \\ &= 1.3_0 \text{ g./cc.} \end{aligned}$$

It follows that the specific heat 'c' of the celeron is given by

$$\left(1 - \frac{3}{2} \times 2.54 \times .096\right) \times c = \frac{3 \times 45.9}{1.3 \times 76}$$

$$\times \left(\frac{0.18}{5} + 0.096\right) = 1.40 \times (0.036 + 0.096)$$

$$= 0.18_5 \pm 0.02,$$

whence

$$c = \frac{0.185 \pm 0.02}{1 - 0.366} = 0.29 \pm 0.03 \text{ calories/g./°C.}$$

A slightly different value of 'c' is obtained by fitting the more exact equation (8a) to the experimental curve at 1.6 inches thickness, yielding $c = 0.28$, and the mean of the two is to be compared with the calorimetrically determined value of 0.33 ± 0.03.

5. Conclusions

Finally, we give in the Table 2 the corrected values of $K \times 10^3$ calculated from equation (7) with $c = 0.30$. It is seen that equation (7) is not entirely adequate, there being a small second-order residual variation with d. Nevertheless, the value of K extrapolated linearly to $d = 0$ (Fig. 3) comes out to $(0.46 \pm 0.01) \times 10^{-3}$, which compares well with the previous figure[4] of 0.450×10^{-3}. The mean variation of the individual values about the straight line of Fig. 3 is ± 0.025 and about the means of successive pairs in Table 2 is only ± 0.015 (when the measurement with the 1/4″ disc is given one-half the weight of the others), thus indicating an accuracy of about $\pm 5\%$.

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