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AN ACCURATE RAPID SEMI-DYNAMICAL METHOD FOR THE DETERMINATION OF THE THERMAL CONDUCTIVITIES OF POOR CONDUCTORS

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1. Introduction

One of the simplest techniques of measuring the thermal conductivity of poor conductors is Lees' disc method[1]. Accurate measurements can be made by this method provided that a steady state of heat flow through the sample is attained and a correction is applied for the heat lost from the periphery of the sample. A formula* for this peripheral loss has been developed and tested in an earlier paper,[2] and the present communication deals with the estimation of the equilibrium temperature through an analysis of the rate of approach to equilibrium.

It is found that if an estimate is made of the time, t(1%), that must elapse before the deviation from the equilibrium condition is less than a pre-selected figure of (say) 1%, values of the order of 2 to 10 hours are obtained (cf. section 3 below). The mathematical analysis leads to a rapid method of measuring the thermal conductivity of poor conductors in which the steady conditions are obtained graphically from a number of observations taken during the first half hour of the experiment. The technique has been tested experimentally and gives reliable and accurate results.

2. Theoretical Analysis

In Lees' disc method the sample disc of area 'A' and thickness 'd' is placed between a cylindrical steam-chest at temperature, θ_s, and a metal disc freely suspended in the atmosphere as in Fig. 1 and the conductivity 'K'

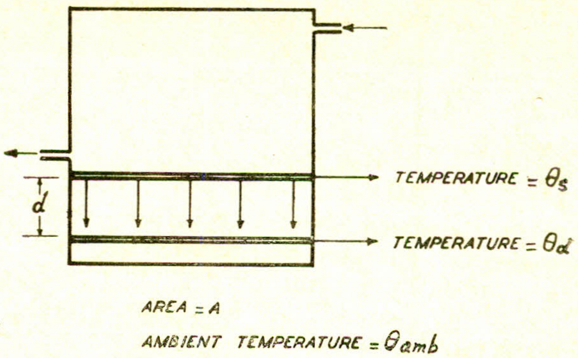

Fig. 1.—Schematic diagram of the experimental arrangement in Lees and Chotlton's disc method for determination of thermal conductivity.

is calculated from the equilibrium temperature, θ_d, of the metal disc as

$$KA\ (\theta_s - \theta_d)/d = E_b\ A_b\ (\theta_d - \theta_{amb}) = E_b'\ A\ (\theta_d - \theta_{amb}) \qquad (1)$$

where θ_m is the ambient temperature of the atmosphere, and the subscript 'b' refers to the brass disc. The flow of heat across a thin sample is governed by the differential equation,

$$\frac{\delta^2\theta}{\delta x^2} = \frac{1}{D}\frac{\delta\theta}{\delta t} = \frac{\rho c}{K}\frac{\delta\theta}{\delta t}, \qquad (2)$$

which we solve by the method of successive approximations, starting with the equilibrium type of distribution, which is linear ($\delta\theta/\delta t = 0$, and therefore $\delta^2\theta/\delta x^2 = 0$) as represented by the line 'PD' in Fig. 2, where the temperature distribution is indicated qualitatively at various stages of the experiment. For small 't' (i.e. before any appreciable quantity of heat flows out into the brass disc), the flow of heat will approximate to that in an infinite bar, giving

$$(\theta_x - \theta_{amb}) = (\theta_o - \theta_{amb})\left(1 - erf\frac{x/2}{\sqrt{Dt}}\right)$$

$$\sim (\theta_o - \theta_{amb})\frac{\sqrt{Dt}}{x}\ exp\left[-\frac{x^2}{4Dt}\right] \qquad (3)$$

$$* \ K = K_{expt} + E\frac{d}{R}\left(K_{expt}/E'_{brass} + \frac{d}{2}\right),$$

where $\dfrac{E}{E'_{brass}} \simeq 0.2$

For large t, on the other hand, the temperature distribution will approximate to the equilibrium-type, being controlled essentially by the heat lost by free radiation from the exposed surface of the metal disc. (The value of t for the transition from one type of distribution to the other can be estimated from (1) and (3) to be

$$t \simeq \frac{d}{D} \div (1/\sqrt{D} + \sqrt{DE'_b/K})^2$$

$$\simeq \frac{\rho c d}{4K} \Big/ \frac{E_b'}{K} \sim 0.1 \times d/E'_b,$$

E_b' being the effective emissivity of the brass disc). The temperature distribution at this stage is given by the full line 'PB' in Fig. 2, and may be assumed as linear to a first approximation :

$$(\theta_s - \theta_x) = \frac{x}{d}(\theta_s - \theta_d),$$

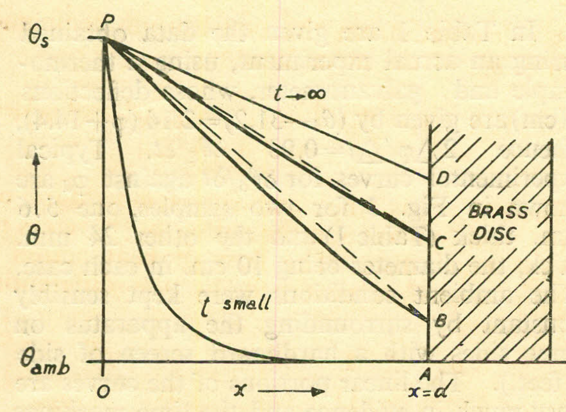

Fig. 2.—Schematic representation of the temperature distribution across the thickness, d, of the specimen at various intervals of time since the placing of the steam chest.

whence

$$\frac{\delta\theta_x}{\delta t} = \frac{x}{d}\,\frac{\delta\theta_d}{\delta t} \tag{4a}$$

and we use this relation to satisfy the differential equation for heat conduction without peripheral loss (equation 2). Thus

$$\frac{\delta^2\theta_x}{\delta x^2} = \frac{\rho c}{K}\,\frac{\delta\theta_x}{\delta t}$$

$$\simeq \frac{\rho c}{K}\,\frac{x}{d}\,\frac{\delta\theta_d}{\delta t}.$$

On repeated integration with respect to x, this gives first

$$\frac{\delta\theta_x}{\delta x} \simeq \left(\frac{\delta\theta}{\delta x}\right)_{x=0} + \frac{\rho c}{K}\,\frac{x^2}{2d}\,\frac{\delta\theta_d}{\delta t},$$

and then

$$\theta_x \simeq \theta_{x=0} + x\left(\frac{\delta\theta}{\delta x}\right)_{x=0} + \frac{\rho c}{K}\,\frac{x^3}{6d}\,\frac{\delta\theta_d}{\delta t}, \tag{4b}$$

whence

$$\frac{\theta_s - \theta_d}{d} = \frac{\theta_{x=0} - \theta_{x=d}}{d} = -\left(\frac{\delta\theta}{\delta x}\right)_{x=0} - \frac{\rho c}{K}\,\frac{d}{6}\,\frac{\delta\theta_d}{\delta t},$$

$$= -\left(\frac{\delta\theta}{\delta x}\right)_{x=d} + \frac{\rho c}{K}\,\frac{d}{3}\,\frac{\delta\theta_d}{\delta t} \tag{5}$$

Equating heat transmitted at $x=d$ to the sum of that absorbed and radiated by the metal disc (of mass m' and Sp. Ht. c') and using equation (5), we get

$$A\,E'_b\,(\theta_d - \theta_{amb}) + c'\,m'\,\frac{\delta\theta_d}{\delta t}$$

$$= -K\,A\left(\frac{\delta\theta}{\delta x}\right)_{x=d}$$

$$= K\,A\left(\frac{\theta_s - \theta_d}{d} - \frac{\rho c}{K}\,\frac{d}{3}\,\frac{\delta\theta_d}{\delta t}\right),$$

whence we have

$$\left(m'\,c' + \frac{\rho A d}{3}\,C\right)\frac{\delta\theta_d}{\delta t}$$

$$= \frac{KA}{d}\,(\theta_s - \theta_d) - A\,E'_b\,(\theta_d - \theta_{amb}) \tag{6a}$$

i.e.

$$(m'\,c' + \tfrac{1}{3}\,m\,c)\,\frac{\delta\theta_d}{\delta t} = \left(\frac{KA}{d}\,\theta_s + A\,E_b'\,\theta_{amb}\right) - \left(\frac{KA}{d} + A\,E_b'\right)\theta_d \tag{6b}$$

Equations (5) and (6) are based on an approximate temperature distribution, and can be refined by starting with the value of $\delta\theta_x/\delta t$ given by the distribution of equation 4(*b*), viz.

$$\frac{\delta\theta_x}{\delta t} = \frac{x}{d}\frac{\delta\theta_d}{\delta t} + \frac{\rho c}{K}\frac{d^2}{6}\left(\frac{x^3}{d^3} - \frac{x}{d}\right)\frac{\delta^2\theta_d}{\delta t^2},$$

which yields

$$\frac{\theta_s - \theta_d}{d} = -\left(\frac{\delta\theta}{\delta x}\right)_{x=o} - \frac{\rho c}{K}\frac{d}{6}\frac{\delta\theta_d}{\delta t} - \left(\frac{\rho c}{K}\right)^2\frac{d^2}{6}\left(\frac{d}{20} - \frac{d}{6}\right)\frac{\delta^2\theta_d}{\delta t^2}$$

$$= -\left(\frac{\delta\theta}{\delta x}\right)_{x=d} + \frac{\rho c}{K}\frac{d}{3}\frac{\delta\theta_d}{\delta t}\left\{1 - \frac{d^2}{15}\frac{\rho c}{K}\frac{\delta}{\delta\theta_d}\left(\frac{\delta\theta_d}{\delta t}\right)\right\}$$

Comparison of this result with equation 5 shows that the present refinement has the effect of multiplying the correction term to the linear temperature distribution by the factor

$$\left\{1 - \frac{d^2}{15}\frac{\rho c}{K}\frac{\delta}{\delta\theta_d}\left(\frac{\delta\theta_d}{\delta t}\right)\right\}$$

$$= \left\{1 - \frac{1}{15} mc \frac{d}{KA}\frac{\partial}{\theta\delta_d}\left(\frac{\delta\theta_d}{\delta t}\right)\right\}$$

$$= \left(1 + \frac{1}{15}\frac{mc}{m'c'} + \ldots\right),$$

so that ultimately the quantity 1/3 mc in equation 6(*b*) will be multiplied by this factor.

3. Experimental Test of the Method

It follows from equations 6 that the graph of $\delta\theta_d/\delta t$ against θ_d is a st. line (for $t > 0.1\ d/E'_b$) and can be accurately exptrapolated to $\delta\theta_d/\delta t = o$ to give the equilibrium value of θ_d. The time required for the actual experimental approach to equilibrium can be estimated by integration of eq. (6), which gives.

$$\ln\left(\frac{\theta_{equilib}}{\theta_{equilib} - \theta}\right) = \frac{KA/d + A\,E'_b}{m'\,c' + \frac{1}{3}\,mc}\,t$$

$$\simeq \frac{KA/d}{\rho' A\,d'\,c'}\,t = \frac{1.3}{d}\frac{K}{d'}\,t.$$

For an accuracy of 1% in the measurement of K, $(\theta_{equilib} - \theta)/\theta_{equilib} \sim \frac{1}{2}\%$, whence $t\,(1\%) \simeq 4d'\,d/K \simeq 3d/K$ seconds

$$= \frac{d/1.2}{1{,}000\ K}\ \text{hours},$$

which works out to seven hours for 1 inch thick samples of building materials with

$K = 0.3 \times 10^{-3}$ cal sec^{-1} cm^{-1}/°C.

In Table 1 are given the data obtained during an actual experiment, using a thermocouple and galvanometer, whose deflections φ (cm) are given by $(\theta_d - 31.2) = 2.14\,(\varphi + 14.4)$, whence $2\triangle\varphi/\triangle t = 0.93\ \delta\theta_d/\delta t$. Typical experimental curves for $\delta\theta_d/\delta t$ against φ_d are shown in Fig. 3 for two samples, one 5.6 mm. thick (Table 1) and the other 24 mm. thick, the diameter being 10 cm. in each case. (The ambient conditions were kept sensibly constant by surrounding the apparatus on three sides with a hardboard screen of side 2 feet.). The linear portions of the curves are very much in evidence and the time markings along the θ axis show that this simple extrapolation technique can reduce the time required for the completion of the experiment to much less than an hour, even for the thick sample, whereas a good approach to equilibrium would require ten times this period. The graph for the 5.6 mm. sample shows clearly that the linear relationship holds to an accuracy of 0.1°C ($\triangle\varphi = 0.05$ cm.) right up to the equilibrium temperature.

Two special features of the method are that (a) no extra formulae are needed for the calculations, which are shown in Table 1 for the 5.6 mm. sample and (b) the extra-

TABIL 1.—THERMAL CONDUCTIVITY MEASUREMENT OF $\frac{1}{4}$ IN. ASBESTOS CEMENT
($K = 0.46 \times 10^{-3}$ cal. sec^{-1} cm^{-1}/°C).

Thickness of the sample = 0.56 cm. Temperature of the cold junction = 31.2°C.

Zero of the Galvonometer = —14.4 cm. Ambient temperature = 32.4° ± 0.2°C.

Time t	Deflection of Galvo = φ	$\Delta\varphi/\Delta t$ ($\Delta t = 1'$)	Mean $2\Delta\varphi/\Delta t = 0.93\delta\theta_d/\delta t$
0′ 0″	—14.18		
0′ 30″	—13.80	1.43	
1′ 0″	—12.75	2.45	3.88
1′ 30″	—11.35	2.80	5.25
2′ 0″	—9.95	2.77	5.57
2′ 30″	—8.58	2.65	5.42
3′ 0″	—7.30	2.53	5.18
3′ 30″	—6.05	2.30	4.83
4′ 0″	—5.00	2.25	4.55
4′ 30″	—3.80	2.15	4.40
5′ 0″	—2.85	1.85	4.00
5′ 30″	—1.95	1.75	3.60
6′ 0″	—1.10	1.60	3.35
6′ 30″	—0.35	1.50	3.10
7′ 0″	+0.40	1.40	2.90
7′ 30″	+1.05	1.35	2.75
8′ 0″	+1.75	1.27	2.62
8′ 30″	2.32	1.07	2.34
9′ 0″	2.82	1.00	2.07
9′ 30″	3.32	0.98	1.98
10′ 0″	3.80	0.88	1.86
10′ 30″	4.20	0.80	1.68
11′ 0″	4.60		

Time t	Deflection of Galvo = φ	$2\Delta\varphi/\Delta t$ ($\Delta t = 2'$)	Mean $2\Delta\varphi/\Delta t = 0.93\delta\theta_d/\delta t$
10′ 0″	3.80		
11′ 0″	4.60	1.60	
12′ 0″	5.40	1.30	1.45
13′ 0″	5.90	0.90	1.10
14′ 0	6.30	0.95	0.92
15′ 0′	6.85	0.88	0.91
16′ 0″	7.18	0.61	0.74
17′ 0″	7.46	0.60	0.60
18′ 0″	7.78	0.49	0.54
19′ 0″	7.95	0.38	0.44
20′ 0″	8.16	0.33	0.36
22′ 0″	8.40	0.28	0.31
24′ 0″	8.71	0.22	0.25
26′ 0″	8.84	0.08	0.15
28′ 0″	8.88	0.08	0.08
30′ 0″	9.00		

The equilibrium temperature (corresponding to $\varphi = 9.02$) is 81.4 °C.

NOTE.—The thermocouple calibration curve is linear and $\Delta\theta = 2.14\Delta\varphi$.

polation of the linear graph is both easy and accurate. If the experiment is stopped when $\delta\theta/\delta t$ has fallen to 1/3 of its maximum value, enough points are usually available to draw at least half of the straight line portion of the graph. The extrapolation will, therefore, involve an error of less than the standard deviation of the individual points in the diagram, which is about 0.2°C. for the thin sample and 0.3°C. for the thick one. This accuracy is confirmed by comparison with the actual equiblibrium values, and is as good as the normal accuracy of measuremen t of the equilibrium value of the temperature, θ_d, of the brass disc.

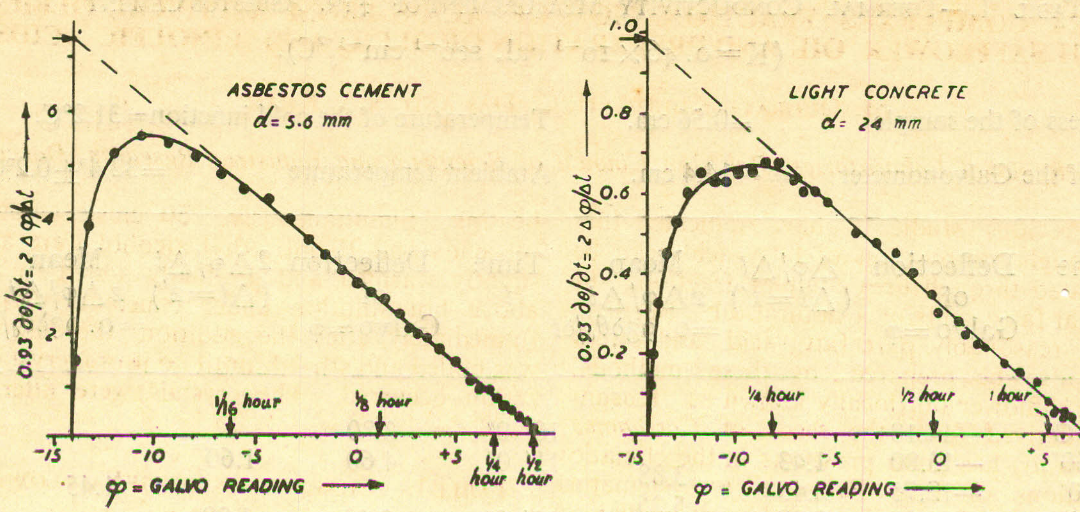

Fig. 3.—Experimental graphs for 0.93 $\delta\theta_d/\delta t$ against θ_d. The abcissa φ are galvanometer readings that vary linearly with the temperature θ_d, which is given by the relation

$\theta_d = 31.2 + 2.14\ (\varphi + 14.4)$, whence $0.93\ \delta\theta_d/\delta t = 2\triangle\varphi/\triangle t$.

In conclusion, it may be noted that extrapolation of the linear portion of the graph in the other direction to $\theta_d = \theta_{amb}$ (i. e. towards $t = 0$) provides data for a fully dynamical method which is being tested experimentally.

References

1. B. L. Worsnop and H. T. Flint *Advanced Practical Physics for students* (Methuen and Co. Ltd., London, 1941), sixth edition, p. 227.
2. A.K.M. Ahsanullah and M.M. Qurashi Pakistan J. Sci. Ind. Research **1**, 136 (1958).