AN INVESTIGATION OF THE CORRECTION FOR PERIPHERAL HEAT-LOSS FROM THICK SAMPLES IN LEES AND CHORLTON'S DISC METHOD OF DETERMINING THERMAL CONDUCTIVITY

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1. Introduction

There are several well-known methods for the determination of the thermal conductivity of poorly conducting building materials like wall-board, light-weight cement, bagasse-board, etc., whose conductivities are in the range of 1×10^{-3} to 0.1×10^{-3} C.G.S. units (cal sec^{-1} cm^{-1}/°C.). Of the various methods, Lees and Chorlton's disc method[1] is perhaps the simplest and easiest to set up and manipulate. The method essentially consists in placing a thin disc of radius 'R' of the sample between a freely-suspended brass disc and a steam chest of the same diameter as the sample (Fig. 1), and then measuring the equilibrium temperature θ_b attained by the brass disc.

The thermal conductivity K of the sample is obtained by assuming that the peripheral surface is small and a uniform temperature gradient exists across the thickness of the sample disc (cf. Fig. 2a). On this basis, the heat flowing per unit time across the faces of the sample disc of thickness 'd' is

$$H = K_{expt} \frac{\theta_s - \theta_b}{d} \times \pi R^2 \qquad (1a)$$

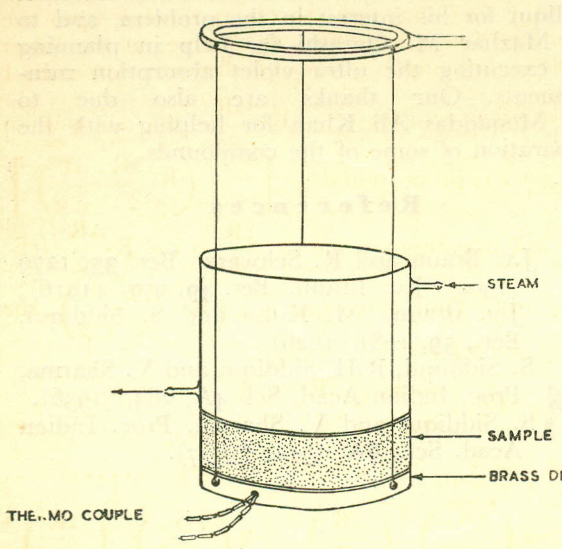

Fig. 1.—Sketch of the experimental arrangement in Lees and Chorlton's disc method for measurement of thermal conductivity of poor conductors.

and this is equated to the heat lost by the brass disc to the surrounding atmosphere (at temperature θ_a), which can be put equal to

$$E_b \times (\pi R^2 + 2\pi R d_b) \times (\theta_b - \theta_a) \qquad (1b)$$

where d_b is thickness of the brass disc, and E_b is a constant according to Newton's law of cooling. However, the application of equations (1) is handicapped by the fact that most of the above-mentioned building materials are only available in fair thicknesses, of the order of 0.5 cm. to 2 cm., and therefore a very considerable loss of heat occurs from the peripheral exposed surface of the specimen. For a sample disc 10 cm. in diameter and 1 cm. thick, this peripheral surface has an area of over 30 sq. cm., which is almost one half the area of the surface across which the conduction of heat occurs. It is therefore of great importance to derive a correction for this peripheral loss.

2. The Basic Pattern of Heat-Flow

The calculation for this effect can be made fairly simply, provided the conditions are such that the peripheral loss is smaller than the heat flowing across the sample face in contact with the brass disc. When this condition is satisfied, we can consider the phenomenon as being essentially one of rectilinear conduction with a small superposed distortion of the lines of heat flow caused by the peripheral heat loss. The nature of this distortion is readily found if we remember that unit area on the periphery of the sample disc loses heat to the surroundings at the rate of $E(\theta - \theta_a)$, where θ is the temperature of the area considered and E is a constant according to Newton's law of cooling. This quantity of heat must naturally flow outwards by conduction from just inside the periphery. The actual heat flow near the periphery is thus the resultant of this outward flow and the normal conduction flow parallel to the axis, whence it follows that the actual lines of flow will be inclined outwards from the axis. Also, at points far from the periphery of the sample disc, the lines of flow will be practically axial, so that the inclination of the lines of flow is a function of the distance from the axis of the disc.

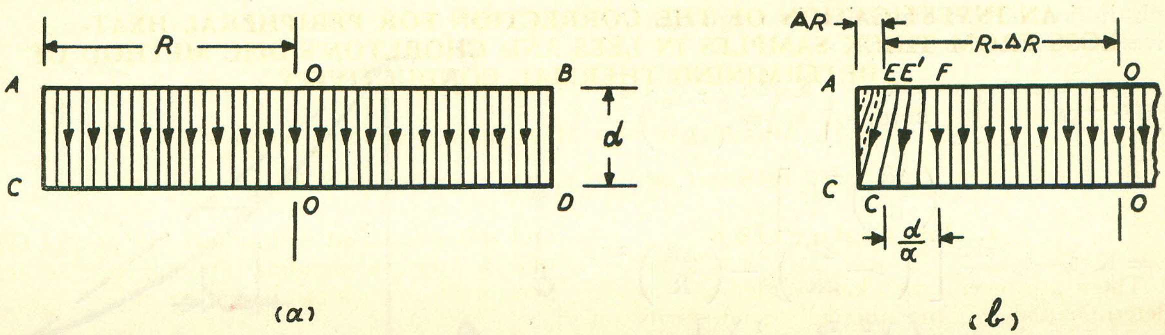

Fig. 2.—Lines of heat flow through the sample disc, assuming (a) ideal case of no peripheral loss, and (b) significant peripheral loss.

It is shown in Appendix I that this inclination is appreciable only within a distance $d/\sqrt{2}$ from the periphery of a disc of thickness 'd'. The ideal and the actual lines of flow are shown schematically in Figs. 2a and 2b, wherein all the lines of flow are parallel to the axis in the immediate neighbourhood of either face of the sample in consequence of the temperature uniformity imposed by the brass disc and the steam-chest. It is seen that only the heat flowing into a circle of radius (R-Δ R) flows out at the bottom of the sample, while the heat flowing into the annulus between radii R and R-Δ R ultimately flows out at the periphery of the sample. This feature enables us to calculate the corrections to be made for the peripheral loss. The correction consists of two components : (a) the peripheral loss as such, and (b) the discrepancy caused by the increased length of the tubes of flow due to their inclination to the axis.

3. Derivation of the Correction Formula

Figure 3a shows an enlarged view of a typical tube of flow, and, if we neglect the small curvature near either face of the sample, this tube can be replaced to a sufficient accuracy by the inclined straight lines of Fig. 3b, in which the undistorted axial flow is shown by the vertical broken lines. If the area intercepted by the inclined tube on the median plane XY is δA, then the mean cross-sectional area of this tube is δA

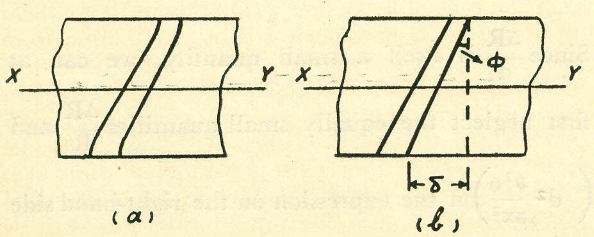

Fig. 3.—Enlarged views of inclined tubes of heat flow (a) exact shape, and (b) approximation suitable for calculations.

$\times \cos\phi$ and its length is $d \sec\phi$, where ϕ is the inclination. Therefore the quantity of heat, δH, flowing per unit time through this tube is given by

$$\delta H = K(\theta_s - \theta_b)\,\frac{\delta A \cos\phi}{d \sec\phi} = \frac{K}{d}(\theta_s - \theta_b) \times \delta A \cos^2\phi$$

$$= \frac{K}{d}(\theta_s - \theta_b) \times \delta A/(1 + \delta^2/d^2), \qquad (2)$$

where θ_s is the temperature of steam, θ_b is the temperature of brass disc, K is the thermal conductivity of the sample, and $\delta = d \tan\phi$ is the displacement of one end of the tube of flow relative to the other. It follows that the total heat H flowing out at the bottom of the sample is

$$H = \int \delta H = \frac{K}{d}(\theta_s - \theta_b) \int \delta A/(1 + \delta^2/d^2)$$

$$= \frac{K}{d}(\theta_s - \theta_b) \int_0^{R - \frac{1}{2}\Delta R} 2\pi r\,dr/(1 + \delta^2/d^2), \qquad (3)$$

where the upper limit of integration is determined by the fact that only the heat flowing upto the inclined line CE in Fig. 2b emerges at the lower face of the sample. Now it is shown in Appendix I that δ/d is zero for $r < (R - \frac{\Delta R}{2} - d/\sqrt{2})$, and for larger r is proportional to $\left[r - \left(R - \frac{\Delta R}{2} - \frac{d}{\sqrt{2}} \right) \right]$ with a maximum value of $\frac{\Delta R}{d}$ at $r = R - \frac{\Delta R}{2}$ (cf. Fig. 2b). Thus if we put $r_0 = \left(R - \frac{\Delta R}{2} - \frac{d}{\sqrt{2}} \right)$, then we have for $r > r_0$,

$$\delta/d = \frac{r - r_0}{d/\sqrt{2}} \times \frac{\Delta R}{d},$$

and $\frac{1}{1 + \delta^2/d^2} = 1 - \delta^2/d^2 + \delta^4/d^4 \ldots\ldots\ldots\ldots$

$$= 1 - 2\left(\frac{r - r_0}{d}\right)^2 \left(\frac{\Delta R}{d}\right)^2 + 4\left(\frac{r - r_0}{d}\right)^4 \left(\frac{\Delta R}{d}\right)^4 \ldots\ldots\ldots\ldots\ldots\ldots$$

Substitution of this expression into equation (3) gives to sufficient accuracy

$$H=K\,\frac{\theta_s-\theta_b}{d}\left[\int_0^{R-\frac{1}{2}\Delta R} 2\pi r\,dr-\int_0^{R-\frac{1}{2}\Delta R} 4\pi r\left(\frac{r-r_0}{d}\right)^2\times\left(\frac{\Delta R}{d}\right)^2 dr\right]$$

$$=K(\theta_s-\theta_b)\,\frac{\pi R^2}{d}\left[\left(1-\frac{\Delta R}{2R}\right)^2-\left(\frac{\Delta R}{R}\right)^2\times\left(\frac{\sqrt{2}}{3}\,\frac{r_0}{d}+\frac{1}{4}\right)\right]$$

$$\simeq K(\theta_s-\theta_b)\,\frac{\pi R^2}{d}\left(1-\frac{\Delta R}{2R}\right)\left(1-\frac{\Delta R}{2R}-\frac{\Delta R}{R}\times\frac{\Delta R}{d}\times\frac{\sqrt{2}}{3}\right) \quad (4)$$

It remains now to obtain an estimate of the quantity ΔR. We notice from Fig. 2b that all the flow lines to the left of the line EC terminate on the periphery of the sample disc, which means that all the heat flowing in at the annulus AE of the disc is to be equated to the heat lost from the periphery, which latter is given by

$$2\pi R\times d\times E\times(\bar{\theta}-\theta_a), \quad (5)$$

where θ_a is the ambient temperature and $\bar{\theta}$ is the mean temperature of the peripheral surface. This mean temperature will be somewhat less than $\frac{1}{2}(\theta_s+\theta_b)$ because the temperature distribution is not truly linear, but is changed by the surface loss into the form shown in Fig. 4. Assuming a linear variation of $\partial\theta/\partial x$ (which is close to the truth), we may put

$$\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial x}+\frac{\partial^2\theta}{\partial x^2}\left(x-\frac{d}{2}\right)=\frac{\theta_b-\theta_s}{d}+\frac{\partial^2\theta}{\partial x^2}\left(x-\frac{d}{2}\right),$$

which integrates to

$$\theta=\theta_s+\frac{\theta_b-\theta_s}{d}\,x+\frac{1}{2}\,\frac{\partial^2\theta}{\partial x^2}\,x\,(x-d) \quad (6a)$$

and gives finally

$$\bar{\theta}=\frac{1}{d}\int_0^d \theta\,dx=\frac{1}{2}(\theta_s+\theta_b)-\frac{d^2}{12}\,\frac{\partial^2\theta}{\partial x^2} \quad (6b)$$

Thus the heat loss from the peripheral surface of the sample becomes

$$2\pi R\times d\times E\left(\frac{1}{2}(\theta_s+\theta_b)-\theta_a-\frac{d^2}{12}\,\frac{\partial^2\theta}{\partial x^2}\right), \quad (7)$$

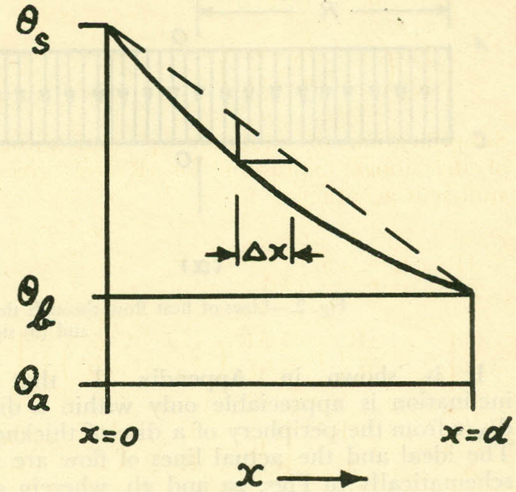

Fig. 4.—Typical temperature distribution on the peripheral surface of the sample.

while the heat flowing into the annulus AE is to a good approximation given by

$$K\,\pi\left(R^2-(R-\Delta R)^2\right)\left(-\frac{\partial\theta}{\partial x}\right)_{x=0}$$

$$=2\pi RK\times\Delta R\left(1-\frac{\Delta R}{2R}\right)\left(\frac{\theta_s-\theta_b}{\sqrt{d^2+(\Delta R)^2}}+\frac{d}{2}\,\frac{\partial^2\theta}{\partial x^2}\right)$$

$$=2\pi R\,\frac{K}{d}\,\Delta R\left(1-\frac{\Delta R}{2R}\right)\left(\frac{\theta_s-\theta_b}{\sqrt{1+(\Delta R/d)^2}}+\frac{d^2}{2}\,\frac{\partial^2\theta}{\partial x^2}\right) \quad (8)$$

On equating the two expressions (7) and (8), and transposing, we get for the maximum inclination of the flow lines

$$\phi_{max}=\frac{\Delta R}{d}=d\,\frac{E}{K}\times\frac{\frac{1}{2}(\theta_s+\theta_b)-\theta_a-\frac{d^2}{12}\partial^2\theta/\partial x^2}{(\theta_s-\theta_b)(1-\frac{1}{2}(\Delta R)^2/d^2)+\frac{1}{2}\partial^2\theta/\partial x^2}\Big/\left(1-\frac{\Delta R}{2R}\right) \quad (9)$$

Since $\frac{\Delta R}{d}$ is itself a small quantity, we can at first neglect the equally small quantities $\frac{\Delta R}{R}$ and $\left(d^2\frac{\partial^2\theta}{\partial x^2}\right)$ in the expression on the right-hand side of equation (9). We thus get to a good approximation,

$$\phi_{max} = \frac{\Delta R}{d} = d\,\frac{E}{K}\left[\tfrac{1}{2}(\theta_s+\theta_b) - \theta_\alpha\right]/(\theta_s-\theta_b)$$

$$= d\,\frac{E}{K}\left(\frac{1}{2} + \frac{\theta_b-\theta_\alpha}{\theta_s-\theta_b}\right) \qquad (10)$$

Simplification of equation (4) and substitution of the above expression for ΔR now gives to a sufficient accuracy

$$H = K(\theta_s-\theta_b)\,\frac{\pi R^2}{d}$$

$$\times\left[1 - \frac{\Delta R}{R}\left(1 + \frac{\Delta R}{d}\left(\frac{\sqrt{2}}{3} - \frac{d}{4R}\right)\right)\right]$$

$$= (\theta_s-\theta_b)\,\frac{\pi R^2}{d}\left[K - \frac{E}{R}\,d^2\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right)\right.$$

$$\left.\times\left(1 + \frac{E}{2K}\,d\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right)\left(\frac{\sqrt{2}}{1.5} - \frac{d}{2R}\right)\right)\right]$$

and finally, $K_{expt} = \frac{H}{\theta_s-\theta_b} \times \frac{d}{\pi R^2}$

$$= K - \frac{E}{R}\,d^2_{reduced}, \qquad (11a)$$

with $d^2_{reduced} = \left(1 + \frac{E}{2K}d\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right)\right.$

$$\left.\times\left(\frac{\sqrt{2}}{1.5} - \frac{d}{2R}\right)\right)\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right)d^2$$

$$= (1+\mu)\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right)d^2 \qquad (11b)$$

where μ is nearly constant and less than 0.1, cf. Table 2. If the neglected quantities, $\frac{\Delta R}{R}$ and $d^2\,\frac{\partial^2\theta}{\partial x^2}$, are also taken into account, then we find in Appendix II that the factor $(1+\mu)$ is to be replaced by $(1-\mu')$, where μ' is again nearly constant and ~ 0.1. μ(or μ') and therefore $d^2_{reduced}$ can be easily calculated provided we have an approximate value of E, which can be got from two experimental determinations of K with different values of the thickness d. The formula (11) then gives us the desired relation between the actual value of K and the experimentally determined value. The correction to the experimental value of K is seen to be inversely proportional to the radius R of the sample and approximately proportional to the square of its thickness, d. This is as it should be, because the dependence on d is two-fold, the peripheral loss increases linearly with d, while the heat conduction across the disc is inversely proportional to d. The influence of the factors $\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right)$, however, needs some elucidation.

4. Behaviour of the Correction Formula

Let us here examine the formula (1a) used for calculating the value of K_{expt} from the equilibrium temperature, θ_b, of the brass disc, wherein it is assumed that there is no peripheral heat loss from the sample. Equations (1) give us

$$K_{expt} = E_b\,\frac{\pi R^2 + 2\pi R d_b}{\pi R^2} \times d\,\frac{\theta_b-\theta_a}{\theta_s-\theta_b}$$

$$= E'_b \times d\,\frac{\theta_b-\theta_a}{\theta_s-\theta_b} \qquad (12)$$

where $E'_b = E_b\,(1 + 2d_b/R)$.

It follows that

$$\left(\frac{1}{2} + \frac{\theta_b-\theta_a}{\theta_s-\theta_b}\right) = \frac{1}{2} + \frac{K_{expt}}{dE'_b}.$$

Substitution of this expression into equation (11b) gives

$$d^2_{reduced} = \left(1 + \left(\frac{\sqrt{2}}{1.5} - \frac{d}{2R}\right)\right.$$

$$\left.\times\left(\frac{1}{2} + \frac{K_{expt}}{d\,E'_b}\right)\frac{Ed}{2K}\right)\left(\frac{1}{2} + \frac{K_{expt}}{dE'_b}\right) \times d^2$$

$$= (1+\mu)\left(\frac{d}{2} \times \frac{K_{expt}}{E'_b}\right) \times d \qquad (13a)$$

$$\text{where } \mu = \left(\frac{\sqrt{2}}{1.5} - \frac{d}{2R}\right)\left(\frac{d}{2} + \frac{K_{expt}}{E'_b}\right) \times \frac{E}{2K}$$

$$\simeq \frac{1}{3}\,\frac{E}{E'_b} \qquad (13b)$$

Finally we get from equation (11a),

$$K_{expt} = K - \frac{E(1+\mu)}{R}\,d\left(\frac{d}{2} + \frac{K_{expt}}{E'_b}\right) \qquad (14)$$

Inclusion of the higher order terms neglected in equation (10) will as before result in the replacement of μ by $-\mu'$, where μ and μ' are both ~ 0.1.

For thin samples, *i.e.*, when d is small, equation (14) reduces to

$$K_{expt} = K - \frac{E}{R} d \frac{K_{expt}}{E'_b}\left(1 + \frac{1}{3}\frac{E}{E'_b}\right)$$

$$\simeq K\left(1 - \frac{d}{R} \times \frac{E}{E'_b}\right) \quad (15)$$

which shows that with the experimental arrangement of Fig. 1, the values of K determined for thin samples are too low by a factor of $\left(1 - \frac{d}{R} \times \frac{E}{E'_b}\right)$.

This is a direct consequence of the fact that even for the thinnest samples, the maximum inclination of the lines of heat-flow does not become zero but rather tends to a constant value. From equations (10) and (12), we have for the inclination at the periphery

$$\left(\phi_{max}\right)_{d \to 0} = \left(\frac{\Delta R}{d}\right)_{d \to 0}$$

$$= \frac{E}{K}\left[d\left(\frac{1}{2} + \frac{\theta_b - \theta_a}{\theta_s - \theta_b}\right)\right]_{d \to 0}$$

$$= \frac{E}{K}\left(\frac{d}{2} + \frac{K_{expt}}{E'_b}\right)_{d \to 0} \simeq \frac{E}{E'_b},$$

which is small in the practical examples discussed below, but is never zero.

In order to study the variation of K_{expt} with the thickness d, we take the following typical values for the constants in equation (14) :

$K = 0.4 \times 10^{-3}$ (cal sec^{-1} cm^{-1}/°C), $E = 0.1 \times 10^{-3}$, $E'_b = 0.3 \times 10^{-3}$, $R = 5$ cm.,

and if we take $K_{expt} \simeq 0.3$, our relation becomes approximately $1000\,K_{expt} = 0.4 - \frac{0.1}{5} d\left(\frac{d}{2} + \frac{0.3}{0.3}\right)$

$$\times\left(1 + \left(0.943 - \frac{d}{2 \times 5}\right)\left(\frac{1}{2} + \frac{0.3}{0.3d}\right) \times \frac{0.1}{2 \times 0.4} d\right)$$

$$= 0.4 - \frac{d}{50}\left(1 + \frac{d}{2}\right)\left(1 + \left(0.943 - \frac{d}{10}\right)\right.$$

$$\left.\times\left(0.125 + \frac{d}{16}\right)\right).$$

The values got from this relation are shown in Table 1 below together with the corresponding values of d and d^2. For small d, the change in K is roughly proportional to d, while for large d, the variation depends more nearly on d^2. If a simple working formula is sought, it is probably accurate enough to try the compromise

$$K_{expt} = K - a\, d^{\frac{3}{2}} \quad (16)$$

where 'a' will be of the order of

$$\frac{E}{R}\left(\frac{K}{E'_b} \times \frac{R \times E}{E'_b}\right)^{\frac{1}{2}} \sim \frac{E}{R}.$$

Such a treatment of the calculated data of Table 1 is shown in Fig. 5 in which the values for d = 0 and 0.5 cm. are taken as unknown. The extrapolation of the straight line to $d^{\frac{3}{2}} = 0$ gives $K = 0.406 \times 10^{-3}$ (cal sec^{-1} cm.$^{-1}$/°C.), which is within $1\frac{1}{2}$ % of the correct value, so that this simplified equation (16) should be acceptable within the usual limits of experimental error for K, *i.e.*, about 0.01×10^{-3} or a little more.

5. Experimental Verification

For experimental work, either this above simplified technique or the more elaborate equation (11) can be used. In order to estimate the relative merits of the two procedures, an analysis is presented for the results of a series of experiments performed in this laboratory with "Celeron" discs of different thicknesses from half an inch upto almost two inches. Several discs 10 cm. in diameter were cut and accurately machined out of a quarter-inch thick sheet of "Celeron," which is a variety of hard laminated insulating material. Composite discs of various thicknesses were made up out of these by putting together 2, 3, 4, etc., of these ¼″ thick discs. The thermal conductivity of these composite discs was determined with the apparatus of Fig. 1 by measuring the equilibrium temperature of the brass disc in each case. The mean results for two sets of determinations are shown in Table 2 along with other relevant data.

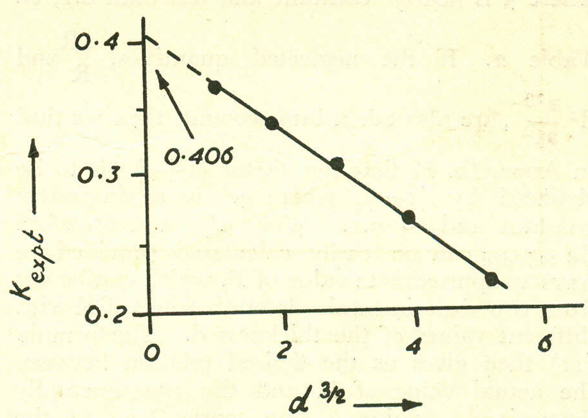

Fig. 5.—Plot of typical theoretical values of K_{expt} against $d^{3/2}$ to show approximate linearity.

TABLE 1.—CALCULATED VARIATION OF K_{EXPT} IN A TYPICAL CASE.

d (cm.) : ..	0	0.5	1.0	1.5	2.0	2.5	3.0
$1000 \times K_{expt}$ ..	0.400	0.38_6	0.36_5	0.33_8	0.30_5	026_5	0.22_0
d^2(cm.²) ..	0	0.25	1.00	2.25	4.00	6.25	9.00

TABLE 2.—EXPERIMENTAL DATA FOR " CELERON " DISCS UPTO 2 INCHES THICK.

"Celeron" Discs : R = 5.0 cm. E'_b for brass disc $= 0.290 \times 10^{-3}$

No. of discs	d (cm.)	θ_b (°c.)	θ_a (°c.)	$\frac{1}{2}+\frac{\theta_b-\theta_a}{\theta_s-\theta_b}$	$\frac{d}{2}+\frac{d(\theta_b-\theta_a)}{\theta_s-\theta_b}$	K_{expt} (cal sec^{-1}cm^{-1}/°c.)	d^2 (cm.²)	$1+\mu$	d^2 reduced	$d^{3/2}$
2	1.27	65.7	29.8	1.55	1.97	$0.40_6 \pm 01_5$	1.61	1.064	2.7	1.4_3
3	1.90	61.2	30.0	1.31	2.49	$0.39_6 \pm 00_8$	3.60	1.07_5	5.1	2.6_2
4	2.54	54.0	29.0	1.05	2.67	$0.38_4 \pm .008$	6.44	1.07_4	7.3	4.0_5
5	3.18	49.4	29.3	0.90	2.86	$0.34_7 \pm .007$	10.1_0	1.07_1	9.7	5.6_6
6	3.81	44.6	28.2	0.80	3.05	$0.31_3 \pm .001$	14.5	1.06_9	12.4	7.4_5
7	4.44	41.9	28.4	0.74	3.28	$0.290 \pm .006$	19.7	1.06_6	15.5	9.3_5

Note :—Experimental values for one disc have been omitted because with our apparatus, the temperature, θ_b, becomes so high that the Newton's law of cooling is no longer accurately applicable to the heat lost by the brass disc.

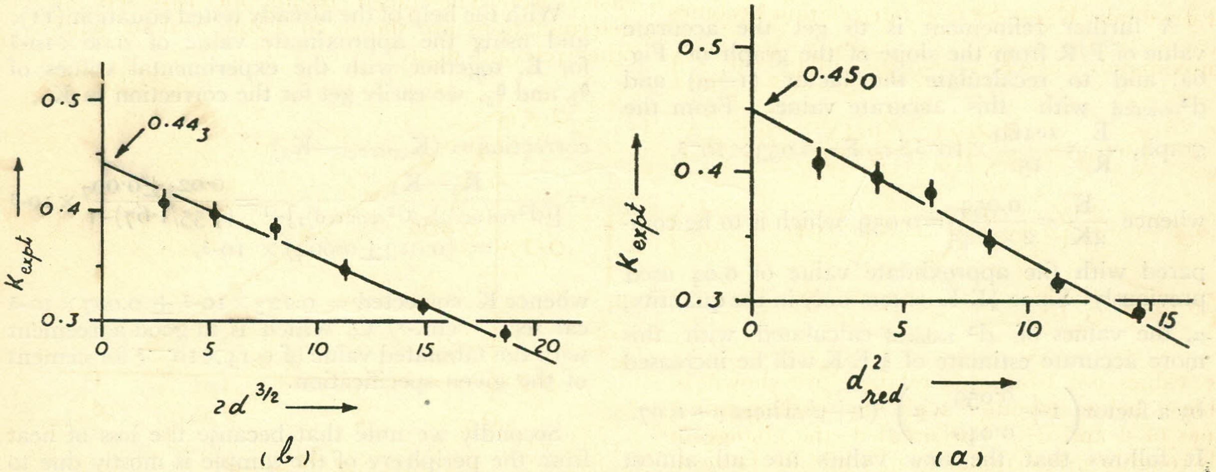

Figure 6. Experimental graphs for "Celeron" discs upto 5 cm. thickness. The measured values (K_{expt}) of theramal conductivity are plotted against (a) d^2reduced, and (b) $d^{3/2}$, and thus extrapolated to zero sample thickness.

First we try to fit to the experimental data, the more accurate equations (11), according to which the graph for K_{expt} against $d^2{}_{reduced}$ should be a straight line. In order to calculate $d^2{}_{reduced}$, we require an approximate value of E (cf. equation 11(b)). This can be done by assuming $d^2{}_{reduced} = d^2$ in the first instance and substituting two of the values from Table 2 into equation 11(a). Thus

$$\left.\begin{array}{l} 0.406 \times 10^{-3} = K - \frac{E}{R} \times 1.61 \\ 0.290 \times 10^{-3} = K - \frac{E}{R} \times 19.7 \end{array}\right\}$$

whence $\frac{E}{R} = \frac{0.116}{18.1} \times 10^{-3} = 0.0064 \times 10^{-3}$,

and therefore $E = 5 \times 0.0064 \times 10^{-3} = 0.032 \times 10^{-3}$

and $\frac{E}{2K} = 0.032/0.8 = 0.04$.

This enables us to calculate the factor $(1+\mu)$ given in column 9 of Table 2 and finally $d^2{}_{reduced}$, which is given in column 10. The graph for K_{expt} against $d^2{}_{reduced}$ is shown in Fig. 6a, in which the short vertical lines through the plotted points indicate their estimated standard errors. The straight line drawn in the figure is seen to fit the experimental points very well, the deviations from the straight line being about equal to the standard errors on the average. This means that equation (11) holds within the limits of experimental accuracy. The corrected value of K found by extrapolation to $d^2{}_{reduced} = 0$ is $0.45_0 \times 10^{-3}$.

A further refinement is to get the accurate value of E/R from the slope of the graph of Fig. 6a, and to recalculate the factor $(1+\mu)$ and $d^2{}_{reduced}$ with this accurate value. From the graph, $\frac{E}{R} = \frac{0{\cdot}160}{15} \times 10^{-3}$, *i.e.*, $E = 0.053 \times 10^{-3}$,

whence $\frac{E}{2K} = \frac{0.053}{2 \times 0{\cdot}45} = 0{\cdot}059$, which is to be compared with the approximate value of 0.04 used previously. Since $\frac{1}{2}E/K$ enters only in the quantity, μ, the values of $d^2{}_{reduced}$ calculated with this more accurate estimate of $\frac{1}{2}$ E/K will be increased by a factor $\left(1 + \frac{0{\cdot}059}{0{\cdot}040} \times \mu\right) / (1+\mu)$, where $\mu \sim 0{\cdot}07$. It follows that the new values are all almost exactly 1.03 times the old ones, so that the graph will be similar to the previous one, but with the abcissa expanded in this ratio. The value of K extrapolated to $d^2{}_{reduced} = 0$ will therefore be the same as before. Similar remarks apply to the replacement of $(1+\mu)$ by $(1-\mu')$ as a further refinement (Appendix II).

For comparison with the foregoing process, we show a plot of the values of K_{expt} against $d^{\frac{3}{2}}$ in Fig. 6b. The points can again be fitted by a straight line within the limits of experimental error, and the value of K obtained by extrapolation to $d = 0$ is $0.44_3 \times 10^{-3}$, which is lower than the corrected value (got from Fig. 6a) by 0.007×10^{-3}, *i.e.*, by about 2%.

6. Examples of Applications and Generalization

We take as an illustration, the evaluation of the correction for peripheral heat loss in a typical case, namely that of a cellular cement disc 10 cm. in diameter and 1.2 cm. thick, the density being 0·35 g./ml. For such a disc, the value obtained experimentally with the apparatus of Fig. 1 was

$K_1 = (K_{expt})_{d=1.2} = 0{\cdot}11_0 \times 10^{-3} \pm 0{\cdot}005 \times 10^{-3}$ cal sec^{-1} cm^{-1}/°C.

Two approaches are now possible. Firstly we may proceed as with the "Celeron" discs, that is, the experimental determination may be repeated with two such cement discs placed one above the other (thus making a single disc of twice the thickness). This gave a value of

$K_2 = (K_{expt})_{d=2{\cdot}4} = 0.08_7 \times 10^{-3} \pm 0{\cdot}005 \times 10^{-3}$ cal sec^{-1} cm.$^{-1}$/°C.

With the help of the already tested equation (11), and using the approximate value of 0.06×10^{-3} for E, together with the experimental values of θ_b and θ_a, we easily get for the correction to K_1,

$$\begin{aligned} \text{correction} &= (K_{correct} - K_1) \\ &= \frac{K_1 - K_2}{[(d^2{}_{reduced})_2/(d^2{}_{reduced})_1] - 1} = \frac{0{\cdot}02_3 \pm 0{\cdot}00_7}{(4{\cdot}55/1{\cdot}67) - 1} \times 10^{-3} \\ &= (0{\cdot}013 \pm 0{\cdot}004) \times 10^{-3}, \end{aligned}$$

whence K corrected $= 0.123 \times 10^{-3} \pm 0.007 \times 10^{-3}$ cal sec^{-1} cm.$^{-1}$/°C, which is in good agreement with the tabulated value of 0.14×10^{-3} for cement of the given specification.

Secondly we note that because the loss of heat from the periphery of the sample is mostly due to convection rather than radiation as such, the value of the constant E in equation (5) will be independent of the nature of the sample. Therefore we can use the experimental value (got from the data on "Celeron" discs) to calculate the correction for any sample with the help of equations (11) or (14). Substitution of the values for E, E_b, and R into equation (14) gives for our apparatus,

$$K_{expt} = K - \frac{0.053 \times 1.1}{5000}\left(\frac{d}{2} + \frac{K_{expt}}{0.29}\right) d,$$

$$= K - \frac{0.0116}{1000}\left(\frac{d}{2} + \frac{K_{expt}}{0.29}\right) d, \qquad (17)$$

if we take $(1+\mu)$ as 1.1 which is quite accurate for our work.* The only quantities needing measurement are d and K_{expt} for the sample disc. Substitution of their values into the above equation gives directly

$$K = 0.11_0 \times 10^{-3} + 0.0116 \times 10^{-3}\left(\frac{1.2}{2} + \frac{0.11}{0.29}\right) \times 1.2$$

$$= (0.11_0 + 0.014) \times 10^{-3} = 0.124 \times 10^{-3}$$
cal sec^{-1} cm.$^{-1}$/°C,

which is in excellent agreement with the previous figure of $0.12_3 \times 10^{-3}$. This agreement provides experimental verification of the fact that, once the apparatus has been calibrated with uniform samples of various thicknesses, the correction for peripheral loss can be directly calculated for an unknown sample from a *single determination* of the thermal conductivity to an accuracy of about 0.01×10^{-3} or better.

Acknowledgement

The authors are indebted to Mr. Sadrul Hasan Rizvi for help with some of the earlier experiments.

Reference

1. B.L. Worsnap and H.T. Flint, *Advanced Practical Physics for Students* (Methuen & Co. Ltd., London, 1941), sixth edition, p. 227.

Appendix I

The Variation of the Inclination, ϕ, of the Tubes of Flow

The precise value of this inclination will depend on the radial distance, r, from the axis of the

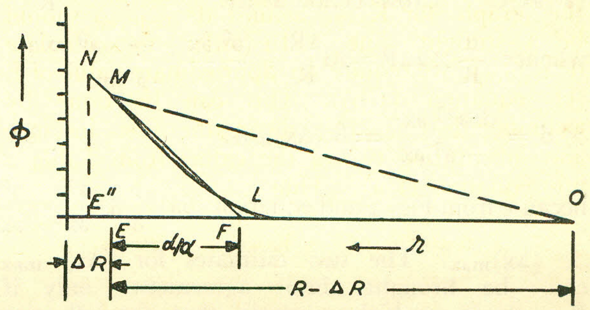

Fig. 7.—Successive approximations to the inclination (ϕ) of the lines of heat flow as a function of the radial distance. First approximation: straight line OM; second approximation: two straight lines, OF and FM; actual: curve OL M.

sample disc, and will be determined by the boundary conditions, *i.e.*, by the heat loss and the temperature gradient on the peripheral surface. An expression for the dependence of ϕ on r can be obtained by successive approximations. Let us assume at first that ϕ is proportional to r, *i.e.*, $\phi = \frac{r}{R}\phi_{max}$, cf. the straight line OM in Fig. 7. Since the isothermal surfaces are normal to the lines of flow, the equation of a section made by the plane of the paper through the isothermal surface for $\theta = \frac{1}{2}(\theta_s + \theta_b)$ will be given by (cf. Figs. 3b & 7)

$$-\frac{dx}{dr} = \tan\phi \simeq \frac{r}{R}\phi_{max} \text{ and } (x)_{r=0} = \frac{d}{2},$$

$$\text{whence } x = \frac{d}{2} - \frac{1}{2}\frac{r^2}{R}\phi_{max},$$

$$\text{and} \quad \Delta x = \left(\frac{d}{2} - x\right) = \frac{1}{2}\frac{r^2}{R}\phi_{max}.$$

Thus the point on the periphery corresponding to the temperature $\frac{1}{2}(\theta_s + \theta_b)$ is shifted from the middle (*i.e.*, x=d/2) by a distance

$$(\Delta x)_{max} = \frac{1}{2}\frac{R^2}{R}\phi_{max} = \frac{1}{2}R\phi_{max} = \frac{R}{d} \times \frac{\Delta R}{2}, \qquad (18a)$$

because $\phi_{max} = \Delta R/d$, cf. Fig. (2). Another estimate of $(\Delta x)_{max}$ can be got by considering the temperature distribution along the tube of flow ECC′ E′ near the periphery (Fig. 2). For this tube,

$$\frac{\text{Corss-section at CC}'}{\text{Cross-section at EE}'} = \left(\frac{R}{R-\Delta R}\right)^2 \simeq 1 + 2\frac{\Delta R}{R}.$$

Now, for any tube, the temperature gradient $\partial\theta/\partial s$ is inversely proportional to the cross-section, so that

*It should be noted that when we determine E from the slope of the graph of Fig. 6 *a*, the value obtained includes any factors that may be introduced by further refinements sueh as those of Appendix II. Thus the replacement of $(1+\mu)$ by $(1-u')$ is automatically taken care of.

$$\frac{(\partial\theta/\partial x)E}{(\partial\theta/\partial x)C} = \frac{\text{Cross-section at CC}'}{\text{Cross-section at EE}'} = 1 + 2\,\frac{\Delta R}{R},$$

whence $\frac{d}{R} \times 2\Delta R = 2d\frac{\Delta R}{R} = \frac{(\partial\theta/\partial x)_C - (\partial\theta/\partial x)_E}{-\partial\theta/\partial x}$

$$\times\, d = \frac{d^2\partial^2\theta/\partial x^2}{-\partial\theta/\partial x} = 8\,(\Delta x)_{max}, \qquad (18b)$$

because from Fig. 4 and equation (6a), $\frac{1}{8}d^2\frac{\partial^2\theta}{\partial x^2} = -\frac{\partial\theta}{\partial x}$ $\times\,(\Delta x)_{max}$. The two estimates for $(\Delta x)_{max}$ can be brought into agreement only if $R = d/\sqrt{2}$, which suggests that the influence of the peripheral loss does not extend all the way inside the disc but is appreciable only to a distance of the order of $d/\sqrt{2}$ inwards from the periphery. This means that the inclination, ϕ, will vary somewhat as shown by the curve OLM in Fig. 7, instead of the straight line OM as initially assumed. For the purposes of calculating the integral of equation (3), a fairly good approximation is got by replacing this curve by the two straight lines OF ($\phi = 0$) and FM, where the distance EF $= d/\alpha$ and α is to be determined from the boundary conditions. Thus the maximum displacement of the isothermal for $\theta = \frac{1}{2}(\theta_s + \theta_b)$ is

$$(\Delta x)_{max} = \frac{d}{2\alpha}\,\phi_{max} = \frac{d}{2\alpha} \times \frac{\Delta R}{d} = \frac{\Delta R}{2\alpha} \qquad (19)$$

For the flow along the tube of flow near EC, we have, analogously to equation (18b), the result

$$2\left[\left(\frac{d/\alpha + \Delta R}{d/\alpha}\right) - 1\right] = 2\,\frac{\Delta R}{d/\alpha} = \frac{8}{d}(\Delta x)_{max},$$

$$\text{whence } (\Delta x)_{max} = \frac{\alpha}{4} \times \Delta R \qquad (20)$$

By equating equations (19) and (20), we get

$$\frac{1}{2\alpha} = \frac{\alpha}{4}, \quad \textit{i.e.}\ \alpha = \sqrt{2}.$$

$$\text{Thus the distance EF} = \frac{d}{\alpha} = d/\sqrt{2}. \qquad (21)$$

Appendix II

Effect of Quantities Neglected in Equation (10)

In order to obtain the departure from linearity of the peripheral temperature distribution of the sample disc, we must calculate the value of $(\Delta x)_{max}$ on the peripheral surface for the middle isothermal, *i.e.*, for $\theta = \frac{1}{2}(\theta_s + \theta_b)$. This isothermal cuts the peripheral surface at $x \sim \frac{d}{2}$, and the line of flow through this intersection will therefore originate near E″, the middle of AE, Figs. 2, 7, at distance $\frac{\Delta R}{2}$ inwards from the periphery. To get the value of $(\Delta x)_{max}$ for this line of flow, we need the variation of ϕ over the region EE″, of extent $\frac{\Delta R}{2}$.

Since all the heat flowing into the annulus AE is ultimately lost to the surroundings by the peripheral surface, whose temperature rises from bottom to top, it is clear that the inclination of the lines of flow will increase as we move from E towards A. The distance EE″ $= \frac{\Delta R}{2}$ being ordinarily much smaller than d, we may, to a first approximation, assume that over this range ϕ continues to increase according to the line MN, which is an extrapolation of the straight line FM. On this basis, we get, as in equation (19),

$$\left[(\Delta x)_{max}\right]_{periphery} = \frac{1}{2}\left(\frac{d}{\sqrt{2}} + \frac{\Delta R}{2}\right)$$

$$\times\left[\phi_{max} \times \frac{d/\sqrt{2} + \Delta R/2}{d/\sqrt{2}}\right]$$

$$= \frac{\Delta R}{2\sqrt{2}}\left(1 + \frac{\Delta R}{d\sqrt{2}}\right)^2 \sim \frac{\Delta R}{2\sqrt{2}}\left(1 + \sqrt{2}\frac{\Delta R}{d}\right)$$

It follows from equation 6(a) that

$$\left(\frac{d^2}{2}\frac{\partial^2\theta}{\partial x^2}\right)\Big/\left(\theta_s - \theta_b\right) = \frac{4}{d}\left[(\Delta x)_{max}\right]_{periphery}$$

$$= \sqrt{2}\,\frac{\Delta R}{d}\left(1 + \sqrt{2}\frac{\Delta R}{d}\right) \qquad (22)$$

Substitution of this result into equation (9) gives us

$$\frac{\Delta R}{d} = \phi_{max} = \frac{\mathbf{E}}{K}d$$

$$\times \frac{\frac{1}{2}(\theta_s + \theta_b) - \theta_o - (\theta_s - \theta_b)\frac{\sqrt{2}}{6}\frac{\Delta R}{d}\left(1 + \sqrt{2}\frac{\Delta R}{d}\right)}{(\theta_s - \theta_b)\left(1 - \frac{1}{2}\left(\frac{\Delta R}{d}\right)^2 + \sqrt{2}\frac{\Delta R}{d}\left(1 + \sqrt{2}\frac{\Delta R}{d}\right)\right)}$$

$$\div \left(1 - \frac{\Delta R}{2R}\right)$$

$$= d\,\frac{E}{K}\left(\frac{1}{2} + \frac{\theta_b - \theta_o}{\theta_s - \theta_b} - \frac{\sqrt{2}}{6}\frac{\Delta R}{d}\left(1 + \sqrt{2}\frac{\Delta R}{d}\right)\right)$$

$$: \left(1+\frac{3}{2}\left(\frac{\Delta R}{d}\right)^2+\sqrt{2}\frac{\Delta R}{d}-\frac{\Delta R}{2R}\right)$$

$$\simeq d\frac{E}{K}\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)\left(\left(1-\frac{\sqrt{2}}{4}\frac{\Delta R}{d}+\frac{1}{2}\left(\frac{\Delta R}{d}\right)^2\right)\right.$$

$$\left.\times\left(1-\sqrt{2}\frac{\Delta R}{d}+\frac{1}{2}\left(\frac{\Delta R}{d}\right)^2+\frac{\Delta R}{2R}\right)\right)$$

$$\simeq d\frac{E}{K}\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)\left(1-1.77\frac{\Delta R}{d}+\frac{3}{2}\left(\frac{\Delta R}{d}\right)^2+\frac{\Delta R}{2R}\right)$$

$$=d\frac{E}{K}\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)(1-\varepsilon) \tag{23}$$

Comparison of this expression with equations (10) and (11) then gives after a little algebra

$$d^2_{\text{reduccd}}=\left(1+\frac{Ed}{2K}(1-\varepsilon)\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)\left(\frac{\sqrt{2}}{1.5}-\frac{d}{2R}\right)\right)\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)(1-\varepsilon)d^2,$$

which simplifies sufficiently accurately to

$$d^2_{\text{reduced}}=\left(1-\frac{Ed}{2K}\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)\left(2{\cdot}6-\frac{d}{2R}\right)\right)\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)d^2$$

$$=(1-\mu'')\left(\frac{1}{2}+\frac{\theta_b-\theta_o}{\theta_s-\theta_b}\right)d^2 \tag{24}$$

where μ' is approximately independent of d and is of the order of 0.1.